Uncovering Mild Drift in Asset Prices with Intraday High-Frequency Data*

Shuping Shi† and Peter C. B. Phillips††

†Macquarie University

††Yale University, University of Auckland & Singapore Management University

April 4, 2024

Abstract

Asset prices are commonly represented as a drift-diffusion process, wherein the drift component denotes the anticipated return of the asset within some time frame, while the diffusion component accommodates random shocks. The drift component has substantial practical significance but accurate estimation is typically challenging and has met with limited success in the existing literature except over large time spans. This paper explores a comprehensive range of drift-diffusion models, encompassing constant, linear, trending, and bursting drift. The investigation aims to identify conditions under which the realized drift (RD) estimator, defined as a normalized realized autocovariance and derived from high-frequency intraday data, can effectively capture integrated drift. The findings indicate that, across all model specifications, RD fails to capture drift at the first order when using data over a fixed time span (T) with a diminishing sampling interval ($\Delta_n \to 0$). However, as the time span extends, the drift component gradually emerges from RD, eventually dominating the diffusion component. Consequently, RD proves to be a reliable tool for gauging integrated drift when the time span is sufficiently extensive. Further, the recently introduced drift-robust quarticity estimator RiceQ is observed to maintain consistency within the double asymptotic framework ($T \to \infty$ and $\Delta_n \to 0$) subject to certain constraints on the divergence rate of T in the presence of various drift forms. With these insights we propose an inferential method to assess the presence of nonzero drift, utilizing RD and RiceQ, and show that the zero drift test has excellent size and power in simulation settings. To illustrate our methodology we investigate drift deviations in the Nasdaq composite index at daily, weekly, fortnightly, and monthly intervals using intraday data.

1 Introduction

Financial asset prices are often represented as continuous time processes with drift and diffusion components. The drift measures deviations of asset prices from a stochastic trend and the diffusion term captures market volatility. Intraday data has made it possible to estimate the latent volatility process with great accuracy using realized measures, as is well documented in previous studies (e.g., Andersen and Bollerslev (1998); Barndorff-Nielsen and Shephard (2004); Barndorff-Nielsen et al. (2009)). At a practical level, realized measures of volatility have proved helpful in reducing option pricing errors, improving investment decisions, measuring risk and uncertainty, and identifying as well as estimating model

*Phillips acknowledges research support from the Kelly Fund at the University of Auckland. Shuping Shi, email: shuping.shi@mq.edu.au. Peter C. B. Phillips, email: peter.phillips@yale.edu.

parameters.[1] Further, intraday data has led to the discovery of other empirical features of asset prices, including jumps (Andersen et al., 2007; Lee and Mykland, 2008), diurnal patterns in volatility (Taylor and Xu, 1997; Andersen and Bollerslev, 1997), and the presence of market microstructure noises (Aït-Sahalia et al., 2005; Ait-Sahalia and Yu, 2009).

Despite its fundamental role in governing asset price dynamics and rewarding investment, drift has not received as much attention in the literature as other empirical features. But several recent studies have emphasized its importance in understanding and modeling asset prices: Christensen et al. (2022) argue that drift forms an integral part of price dynamics across currencies, fixed income investments, equities, and commodities; Laurent and Shi (2020) demonstrate that ignoring drift can lead to significant finite sample bias in realized volatility and severe size distortion in jump testing; and Laurent, Renò, and Shi (2022, LRS hereafter) demonstrate the usefulness of a realized drift measure in forecasting volatility.

Drift measures the expected return of a financial assets and is widely regarded as a challenging quantity to estimate (Merton, 1980; Jagannathan and Ma, 2003). Bandi and Phillips (2003, BP hereafter) argue that estimating drift without a sufficiently long time period is impossible, even with ultrahigh frequency data, and propose a nonparametric estimator for drift. They show that this estimator is only consistent under a double asymptotic setting, where the time span T goes to infinity and the sample interval Δ_n shrinks to zero. In contrast, LRS demonstrate that the drift component can be estimated using realized autocovariance (RAC) within a fixed time span. Under an infill setting, where $\Delta_n \to 0$ and T is fixed, RAC can reveal drift at the first order. The results of BP and LRS are not contradictory but instead depend on the underlying data generating process (DGP) assumed. The DGP of LRS allows for the possibility of extreme drift episodes (captured by an exploding drift coefficient), whereas the standard drift-diffusion process considered in BP does not. The RAC measure becomes revealing under an infill setting when drift explosion occurs at a fast rate but not when drift behavior is mild as in the standard drift formulation considered in BP).

This paper explores the possibility of employing a realized drift (RD) measure based on a normalized version of RAC for the purpose of estimating integrated drift (ID) within a range of drift diffusion processes. These processes encompass the standard drift diffusion process discussed in BP, an explosive linear drift process frequently employed to capture the dynamics of asset prices in the presence of bubbles, a trending drift process, and the drift bursting process introduced in LRS.[2] Compared with the nonparametric estimator put forth by BP, RD offers simplicity in computation and is free from nuisance parameters. As expected, when the sampling interval Δ_n approaches zero while keeping T fixed, RD becomes dominated by volatility and fails to reveal drift at the first order level. However, as the time span extends, the drift component gradually becomes more apparent within RD, ultimately overshadowing the volatility component. Consequently, as T extends towards infinity at a sufficiently fast rate, RD can consistently estimate ID. The specific condition for the divergence rate of T varies depending on the underlying data generation process. Furthermore, the limit distribution follows a normal mixture with conditional variance being the integrated quarticity IQ under the double asymptotic scheme (i.e., $T \to \infty$ and $\Delta_n \to 0$).[3]

To facilitate inference, we consider two estimates of the asymptotic variance IQ: realized quarticity (RQ) and the drift robust quarticity estimator RiceQ proposed by LRS. The infill asymptotics of RQ are well-established, while the consistency of RiceQ under infill conditions is provided by LRS. However, the double asymptotics of these two measures have yet to be explored. We demonstrate that the drift-robust

[1] See Christoffersen et al. (2014); Fleming et al. (2003); Christoffersen and Diebold (2000); Cascaldi-Garcia (2022); Tao et al. (2019), among others.

[2] In contrast to the approach taken in LRS, where a wide range of values is considered for drift and volatility explosion rates, our study focuses on constraining these rates to relatively low magnitudes. This constraint ensures that integrated drift and integrated quarticity can be precisely defined in our analysis.

[3] The infill asymptotics of RAC under the drift bursting process of LRS depends on the specific drift and volatility explosion rates, as detailed in LRS.

quarticity estimator RiceQ remains a consistent estimator for IQ under double asymptotic conditions, given a divergence rate of T that is not too rapid. Additionally, in the presence of drift, the bias of RiceQ is of a smaller order compared to that of RQ.

Furthermore, we introduce an inferential approach to evaluate the existence of non-zero drift, utilizing RD and RiceQ. We demonstrate that this test for non-zero drift is consistent under specific constraints on span parameter T. When there is no drift (under the null hypothesis), the test statistic adheres to standard normal asymptotic behavior, simplifying its practical application. Conversely, in the presence of non-zero drift (alternative hypothesis), the test statistic diverges towards infinity. The rate of divergence is contingent upon the specific data generation process. The detection of nonzero drift suggests the presence of a nonzero expected return and underscores the possibility of arbitrage opportunities.

We simulate data from various drift diffusion processes that account for market microstructure noise, heteroskedasticity, and overnight price jumps, compute RD measures and conduct the zero drift test. The estimation accuracy of RD improves as T rises, as expected. Empirical size of the test is found to be close to nominal in all cases, test power increases with the magnitude of the drift and the length of the time span T. As an empirical illustration, we investigate drift deviations in the NASDAQ composite index. The zero drift null hypothesis is frequently rejected over the sample period, particularly during the dot-com bubble period over 1996–2000. Additionally, we demonstrate that using RiceQ instead of RQ yields significantly sharper inferences concerning RD.

The organization of the paper is as follows: In Section 2, we introduce four model specifications: a standard Ito semimartingale, an explosive linear drift, a trending drift, and drift burst models along with their associated properties. Section 3 presents the realized drift measure and its asymptotic properties within a double asymptotic framework for different data generating processes. In Section 4, we introduce the drift-robust quarticity measure (RiceQ) and outline the conditions necessary for RiceQ to achieve consistency within the double asymptotic framework. Section 5 proposes a drift test and the conditions for its consistency across various data generating processes. Simulation results are reported in Section 5, and Section 6 provides empirical applications, featuring RD measures with confidence intervals for the NASDAQ composite index. Proofs are in the Appendix. The Appendix also includes an exploration of the impact of jumps on both the realized drift estimator and the zero drift test, together with a jump robust version of the zero drift test. An Online Supplement contains further results under infill asymptotic conditions for all model specifications.

Throughout the paper, we assume that there are n equidistant observations spanning the interval $[0, T]$ written in grid form as $\{0 = t_0 < t_1 < \cdots < t_n = T\}$. The distance between two consecutive observations is $\Delta_n = t_i - t_{i-1}$ for all $i \in [1, n]$ so that $T = n\Delta_n$ gives the observation time span.

2 Model Specifications

The log price of a financial asset p_t is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ and is conventionally assumed to follow an Itō semimartingale process of the form

$$p_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \quad \text{for} \quad t \in [0, T], \tag{1}$$

where p_0 is $\mathcal{F}_0$ measurable and W_s is standard Brownian motion. The drift coefficient μ_t and the instantaneous variance of the returns σ_t^2 are assumed to satisfy the following conditions. Integrated drift and integrated quarticity are defined as usual by $\mathsf{ID}_T = \int_0^T \mu_s^2 ds$ and $\mathsf{IQ}_T = \int_0^T \sigma_s^4 ds$.

Assumption 2.1 *Suppose p_t satisfies (1) and there exists a sequence $(T_m)_{m\geq 1}$ of stopping times increasing to infinity and a sequence $(C_m)_{m\geq 1}$ of constants such that for all $m \geq 1$*

(i) $|\mu_t| + |\sigma_t| + \left|\sigma_t^{-1}\right| \leq C_m$ *for all* $t \in [0, T_m]$*;*

(ii) $\mathbb{E}\left[\left|\mu_{t\wedge T_m} - \mu_{s\wedge T_m}\right|^2 + \left|\sigma_{t\wedge T_m} - \sigma_{s\wedge T_m}\right|^2\right] \leq C_m \left|t-s\right|^{\Gamma}$ *for all* $t, s \in [0, T]$*, and* $\Gamma > 0$*; and*

(iii) as $T \to \infty$*,*

$$T^{-1}\int_0^T \mu_s^2 ds \to_{a.s.} \omega_{\mu^2}, \qquad T^{-1}\int_0^T \mu_s^2\sigma_s^2 ds \to_{a.s.} \omega_{\mu\sigma}, \qquad T^{-1}\int_0^T \sigma_s^4 ds \to_{a.s.} \omega_{\sigma^4},$$

where ω_f *represents a random variate related to* f*.*

Assumption 2.1(i) requires the diffusion coefficient to be bounded above and away from zero and the drift coefficient to be bounded. The two coefficients are further required to be smooth in a Hölder continuous sense in Assumption 2.1(ii). Assumption 2.1(iii) clearly holds immediately for strictly stationary and ergodic processes (μ_s, σ_s) but also more widely as a general stability condition. Under Assumption 2.1, the non-arbitrage condition $\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds < \infty$ is satisfied when T is fixed and violated when $T \to \infty$.

The requirements placed on the drift and diffusion coefficients are conventional and encompass a wide range of well-known continuous time processes frequently utilized in financial econometric studies. For example, it includes the constant drift process ($\mu_s = \mu$), the stationary linear drift process, and a random drift process.

Example 2.1 (Stationary Linear Drift Process) The linear drift process is given by

$$p_t = \int_0^t \theta p_s ds + \int_0^t \sigma_s dW_s, \tag{2}$$

where the drift coefficient has the linear form $\mu_s = \theta p_s$ with the coefficient $\theta \leq 0$. Under Assumption 2.1(i) and (ii) and the condition that $\theta \leq 0$, the process is stationary and ergodic and therefore satisfies Assumption 2.1(iii). These properties make the linear process amenable to analysis and it is commonly used in empirical work.

Example 2.2 (Random Drift Process) Suppose that $\mu_t = \ell b_t$ with constant $\ell > 0$ and b_t following a scaled uniform distribution. Specifically

$$p_t = \int_0^t \ell b_s ds + \int_0^t \sigma_s dW_s, \tag{3}$$

where $b_t \sim_{iid} U[-1, \epsilon]$ with $\epsilon < 1$. The drift coefficient μ_t satisfies all three requirements in Assumption 2.1. The Euler discretization of the process gives the random drift martingale process of Phillips and Shi (2019):

$$p_{t_i} = L_{t_{i-1}} + p_{t_{i-1}} + \omega_{t_i}\varepsilon_{t_i} \text{ with } \varepsilon_{t_i} \sim_{i.i.d} \mathcal{N}(0,1) \tag{4}$$

in which $L_{t_{i-1}} = \ell\Delta_n b_{t_{i-1}}$ and $\omega_{t_i} = \sigma_{t_{i-1}}\sqrt{\Delta_n}$. This process was proposed to capture the behavior of asset prices during crisis periods or episodes of financial bubble collapse. The positive scale quantity ℓ measures the random drift shock intensity associated with b_t, which is uniform on an interval ranging from minus unity to a (usually small) positive value ϵ. The random L_t process captures the impact of negative shocks that occur during the crisis or bubble collapsing phase.

While the Itō semimartingale process (1) has been identified as appropriate for modeling log price dynamics under many market conditions, it falls short in accurately capturing dynamics in other market conditions particularly when price behavior is more extreme. As a result several alternative specifications have been introduced, such as the three following examples.

Explosive Linear Drift Process

Suppose that log prices follow a linear drift diffusion process of the form (2) but with $\mu_t = \theta p_t$ and $\theta > 0$, which we refer to as an explosive linear drift process. Under this specification, studied recently in Laurent and Shi (2022), log price is dominated by the linear drift component in such a way that

$$p_t = p_0 \exp\left(\int_0^t \theta dt + \int_0^t \frac{\sigma_s}{p_s} dW_s\right) = p_0 \exp(\theta t)\left[1 + o_p(1)\right], \tag{5}$$

implying that the drift coefficient μ_t which depends on p_t diverges as $t \to \infty$. Then, when the span $T \to \infty$, the drift related quantities have the following asymptotic orders:

$$\begin{aligned} \int_0^T \mu_s^2 ds &= \int_0^T \theta^2 p_s^2 ds = \frac{1}{2}\theta p_0^2 \left(e^{2\theta T} - 1\right)\left[1 + o_p(1)\right] = O_p\left(e^{2\theta T}\right), \\ \int_0^T \mu_s^2 \sigma_s^2 ds &= \int_0^T \theta^2 p_s^2 \sigma_s^2 ds = O_p\left(e^{2\theta T}\right). \end{aligned}$$

Evidently, the explosive linear drift process does not meet the criteria outlined in Assumption 2.1 when $T \to \infty$. The non-arbitrage condition

$$\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds = \int_0^T \left(\frac{\theta p_s}{\sigma_s}\right)^2 ds < \infty$$

is satisfied when T is fixed but violated when $T \to \infty$.

Nevertheless, this specification has significant empirical value due to its close association with the explosive discrete-time model, which is frequently employed to capture the behavior of asset prices during the expansion phase of financial bubbles. Specifically, in periods of bubble expansion, a common empirical formulation for the log prices of financial assets is an explosive discrete time dynamic of the autoregressive form

$$p_{t_i} = \rho_n p_{t_{i-1}} + v_{t_i} \text{ with } \rho_n > 1, \tag{6}$$

with errors v_{t_i} (e.g., Phillips et al. (2011, 2015a,b); Shi and Phillips (2022)). This explosive dynamic is readily derived from the linear drift process using Euler discretization of the continuous process, which yields

$$p_{t_i} = \gamma_n p_{t_{i-1}} + \omega_{t_i - 1}\varepsilon_{t_i} \text{ with } \varepsilon_{t_i} \sim_{i.i.d} \mathcal{N}(0,1),$$

in which the autoregressive coefficient $\gamma_n = 1 + \theta\Delta_n > 1$ and $\omega_{t_i-1} = \sigma_{t_{i-1}}\sqrt{\Delta_n}$.

Trending Drift Process

Suppose the log price is driven by a model with trending drift such that

$$p_t = \int_0^t (\theta_0 + \theta_1 s)\, ds + \int_0^t \sigma_s dW_s, \tag{7}$$

where the drift coefficient $\mu_s = \theta_0 + \theta_1 s$. Like the explosive linear drift process, the non-arbitrage condition

$$\int_0^T \frac{\mu_s^2}{\sigma_s^2} ds = \int_0^T \left(\frac{\theta_0 + \theta_1 s}{\sigma_s}\right)^2 ds < \infty$$

is satisfied when T is fixed and violated when $T \to \infty$. As $T \to \infty$ the asymptotic orders of these two quantities related to drift are

$$\int_0^T \mu_s^2 ds = \int_0^T (\theta_0 + \theta_1 s)^2\, ds = O\left(T^3\right) \text{ and } \int_0^T \mu_s^2 \sigma_s^2 ds = O\left(T^3\right).$$

This behavior again fails to align with Assumption 2.1 when $T \to \infty$ and discretization using the Euler method results produces a random walk with a linear trend process

$$p_{t_i} = \gamma_{n,0} + \gamma_{n,1} t_{i-1} + p_{t_{i-1}} + \omega_{t_{i-1}} \varepsilon_{t_i}, \tag{8}$$

in which $\gamma_{n,0} = \theta_0 \Delta_n$ and $\gamma_{n,1} = \theta_1 \Delta_n$.

Drift Bursting Process

Consider a drift bursting process of the following form

$$p_t = \int_0^t \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds + \int_0^t \sigma_s \left(1 - \frac{s}{T}\right)^{-\beta} dW_s, \text{ for } \ t \in [0, T], \tag{9}$$

where μ_s and σ_s satisfy Assumption 2.1. The respective rates of drift and volatility explosion are controlled by the parameters α and β, which are assumed to be positive constants. This process captures a unidirectional drift explosion of log prices, with T representing the duration and the peak location of this explosion. It includes the standard model in (1) as a special case with $\alpha = \beta = 0$. The non-arbitrage condition requires

$$\int_0^T \left(\frac{\mu_s \left(1 - \frac{s}{T}\right)^{-\alpha}}{\sigma_s \left(1 - \frac{s}{T}\right)^{-\beta}} \right)^2 ds = \int_0^T \frac{\mu_s^2}{\sigma_s^2} \left(1 - \frac{s}{T}\right)^{-2(\alpha-\beta)} ds = T \int_0^1 \frac{\mu_{Tu}^2}{\sigma_{Tu}^2} (1-u)^{-2(\alpha-\beta)} du < \infty,$$

which is satisfied when $\alpha - \beta < \frac{1}{2}$ under the infill asymptotic scheme but violated when $T \to \infty$.

The drift bursting process was initially proposed by Christensen et al. (2022) to capture the dynamic of asset prices in the event of flash crashes, where the log prices of financial assets plummet within a very short window but quickly recover. The drift bursting model (9) is a slightly modified version of those considered in Christensen et al. (2022), Andersen et al. (2021) and Laurent et al. (2022). Here, we allow the drift coefficient $\mu_{T,s} \equiv \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha}$ and the diffusion coefficient $\sigma_{T,s} \equiv \sigma_s \left(1 - \frac{s}{T}\right)^{-\beta}$ to depend on the duration of the burst event T.[4] The relationship between the drift coefficient and T is analogous to a local-to-unity process specification (Phillips, 1987) in which the autoregressive coefficient is influenced by T. When a drift bursting episode occurs over an extended time frame, the expansion is anticipated to be more gradual compared to instances that unfold within a brief time window.

The drift coefficient $\mu_{T,s}$ explodes as t approaches T, with the rate of explosion depending on both α and T, and hence fails the bound requirement of Assumption 2.1. When $\alpha < 1/2$

$$\int_0^T \mu_{T,s}^2 ds = \int_0^T \mu_s^2 \left(1 - \frac{s}{T}\right)^{-2\alpha} ds \leq \mu_{\max}^2 T \int_0^1 u^{-2\alpha} du = O_p(T).$$

The quantity $\int_0^1 u^{-2\alpha} du$ is non-convergent when $\alpha \geq 1/2$ and $\int_0^T \mu_{T,s}^2 ds$ diverges to infinity at a faster rate than T. The Euler discretization of the drift bursting model is given by:

$$p_{t_i} = \mu_{t_{i-1}} \Delta_n \left(1 - \frac{t_{i-1}}{T}\right)^{-\alpha} + p_{t_{i-1}} + \tilde{\omega}_{t_{i-1}} \varepsilon_{t_i} \text{ with } \varepsilon_{t_i} \sim_{i.i.d} \mathcal{N}(0,1),$$

where $\tilde{\omega}_{t_{i-1}} = b_{t_{i-1}} \sqrt{\Delta_n} \left(1 - \frac{t_{i-1}}{T}\right)^{-\beta}$. This discretization contrasts with that of the linear trending drift diffusion process (7) but shares some similarity with the discrete time polynomial trend model of Wang and Yu (2023).

[4] The time period is fixed at $T = 1$ in both Laurent et al. (2022) and Andersen et al. (2021). Additionally, there was no volatility bursting in Andersen et al. (2021); and drift bursting was treated as noise.

Assumption 2.2 *$\alpha \in [0, 1/2)$ and $\beta \in [0, 1/4)$.*

The non-arbitrage condition holds automatically under Assumption 2.2. Condition $\alpha < 1/2$ is imposed so that integrated drift $\mathsf{ID}_T \equiv \int_0^T \mu_s^2 (1 - \frac{s}{T})^{-2\alpha} ds$ is well defined and condition $\beta < 1/4$ ensures well defined integrated quarticity $\mathsf{IQ}_T \equiv \int_0^T \sigma_s^4 (1 - \frac{s}{T})^{-4\beta} ds$.

3 Realised Drift

The return over period $[t_{i-1}, t_i]$ is represented as $r_{n,i} = p_{t_i} - p_{t_{i-1}}$ for $i = 1, 2, \ldots, n$. The realised drift RD_T is defined as the normalized first order autocovariance, i.e.,

$$\mathsf{RD}_T = \frac{1}{\Delta_n} \sum_{i=2}^{n} r_{n,i} r_{n,i-1}.$$

where Δ_n is the time interval between t_{i-1} and t_i. In Li and Linton (2022), the realized autocovariance estimator garnered attention as it was applied to ultra-high-frequency data, with the specific aim of capturing market microstructure noise dependencies. The study conducted by LRS revealed that the realized autocovariance can also effectively function as an estimator for large bursting drift when applied to relatively low-frequency observations such as 5-minute interval data. In both these scenarios the limit properties of the realized drift estimator were developed through the infill asymptotic approach.[5]

We now explore the asymptotic behavior of the realized drift estimator under both the infill and double asymptotic settings across a range of model specifications, including a standard Ito semimartingale process, an explosive linear drift process, a trending drift process, and a drift bursting process.

Asymptotics: Standard Modeling Framework

Under the generating mechanism (1), RD can be decomposed as follows

$$\mathsf{RD}_T = D_{1,n} + D_{2,n} + D_{3,n}, \tag{10}$$

where

$$D_{1,n} = \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \mu_s ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s ds \right), \tag{11}$$

$$D_{2,n} = -\frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \mu_s ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s \right) - \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s ds \right), \tag{12}$$

$$D_{3,n} = \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s \right). \tag{13}$$

The first component $D_{1,n}$ is related to the drift terms, $D_{3,n}$ to the diffusion terms, and $D_{2,n}$ to the cross-products of the drift and diffusion terms. The limit properties of these quantities are given in the following result.

[5] Drawing on infill asymptotic methods, Kolokolov et al. (2023) introduced a block uniformly minimum variance unbiased (BUMVU) estimator for integrated drift. This estimator, in conjunction with a bootstrapping procedure, is used in their work to detect drifts. While our focus in the present work is on RD_T estimator, the same analytical approach can be extended to BUMVU using long span asymptotics.

Lemma 3.1 *Suppose that p_t follows a standard Ito semimartingale process, i.e., (1) with Assumption 2.1. When $T \to \infty$ and $\Delta_n \to 0$,*

$$\begin{aligned}
&(1)\, T^{-1} D_{n,1} \to_p \omega_{\mu^2}, \\
&(2) T^{-1/2} D_{n,2} \xrightarrow{d} Z_3 \equiv \mathcal{MN}\left(0, 4\omega_{\mu\sigma}\right), \\
&(3) n^{-1/2} D_{n,3} \xrightarrow{d} Z_4 \equiv \mathcal{MN}\left(0, \omega_{\sigma^4}\right).
\end{aligned}$$

Recall that under the infill asymptotic scheme when T is fixed,

$$D_{n,1} \to_p \int_0^T \mu_s^2 ds,\ D_{n,2} \xrightarrow{d} Z_1 \equiv \mathcal{N}\left(0, 4\int_0^T \mu_s^2 \sigma_s^2 ds\right), \text{ and } \Delta_n^{1/2} D_{n,3} \xrightarrow{d} Z_2 \equiv \mathcal{N}\left(0, \int_0^T \sigma_s^4 ds\right).$$

The quantity $D_{n,3}$ diverges at the rate $O_p(\Delta_n^{-1/2})$, whereas both $D_{n,1}$ and $D_{n,2}$ are $O_p(1)$ with random limits. The realised drift estimator can be written in asymptotic form as

$$\mathsf{RD}_T \sim_a \underbrace{\Delta_n^{-1/2} Z_2}_{\text{first order}} + \underbrace{\int_0^T \mu_s^2 ds + Z_1}_{\text{second order}}.$$

The realized drift estimator RD_T cannot reveal drift at the first order but shows a noise contaminated integrated drift at the second order under the infill setting.[6] Under the double asymptotic scheme, from Lemma 3.1,

$$\mathsf{RD}_T \sim_a \int_0^T \mu_s^2 ds + T^{1/2} Z_3 + n^{1/2} Z_4.$$

The limit distributions of RD_T follow immediately.

Theorem 3.1 *Suppose p_t follows a standard Ito semimartingale process (1) with Assumption 2.1. Under double asymptotics we have*

$$\mathsf{RD}_T = \begin{cases} D_{n,3}\left[1 + o_p(1)\right] = O_p\left(n^{1/2}\right) & \text{if } T\Delta_n \to 0 \\ \left[D_{n,1} + D_{n,3}\right]\left[1 + o_p(1)\right] = O_p(T) & \text{if } T\Delta_n \to O_+(1) \\ D_{n,1}\left[1 + o_p(1)\right] = O_p(T) & \text{if } T\Delta_n \to \infty \end{cases},$$

which implies: (i) when $T\Delta_n \to 0$,

$$n^{-1/2}\, \mathsf{RD}_T \xrightarrow{d} \mathcal{MN}\left(0, \omega_{\sigma^4}\right);$$

(ii) when $T\Delta_n \to O_+(1)$,

$$\mathsf{RD}_T \sim_a \int_0^T \mu_s^2 ds + n^{1/2} Z_4;$$

and (iii) when $T\Delta_n \to \infty$,

$$n^{-1/2}\left[\mathsf{RD}_T - \int_0^T \mu_s^2 ds\right] \xrightarrow{d} \mathcal{MN}\left(0, \omega_{\sigma^4}\right). \tag{14}$$

[6] The Online Supplement demonstrates that when T is fixed and $\Delta_n \to 0$, RD_T cannot reveal integrated drift under all the models considered in this paper.

Under double asymptotics when $T\Delta_n \to 0$, RD_T is therefore dominated by noise term $D_{n,3}$ which is asymptotically equivalent to $n^{1/2}Z_4$. When the time span expands so that $T\Delta_n = O_+(1)$, RD_T reveals drift heavily contaminated by noise in the first order, viz. $\int_0^T \mu_s^2 ds + n^{1/2}Z_4$. Most interestingly, when the time span T is long enough that $T\Delta_n \to \infty$, RD_T is asymptotically equivalent to the squared magnitude of the drift

$$\mathsf{ID}_T = \int_0^T \mu_s^2 ds$$

over the interval $[0, T]$. In this case the limiting property in (14) can be reformulated as

$$\mathsf{RD}_T \sim_a \mathcal{MN}\left(\int_0^T \mu_s^2 ds,\ \Delta_n^{-1}\int_0^T \sigma_s^4 ds\right), \tag{15}$$

so that RD_T is an asymptotically unbiased estimator of the integrated drift ID_T but with increasing asymptotic variance. In particular, under Assumption 2.1,

$$\Delta_n^{-1}\int_0^T \sigma_s^4 ds = O_p(n) \to \infty, \text{ as } \Delta_n \to 0, T \to \infty. \tag{16}$$

In other words, the realized drift estimate RD_T requires normalization of the difference $\mathsf{RD}_T - \int_0^T \mu_s^2 ds$ for asymptotic stability. In fact, as will be shown in Section 5, self normalization by suitable quarticity estimates produces test statistics with good properties that enable inference about integrated drift ID_T.

Asymptotics: Explosive Linear Drift

In the framework of explosive linear drift the relationship $\mu_s = \theta p_s$ holds, where θ is a positive constant. The results of Theorem 3.2 detail the convergence properties of RD_T, considering both infill and double asymptotic scenarios. The test statistic can again be decomposed into three terms as in (10).

Theorem 3.2 *Suppose p_t follows an explosive linear drift model, i.e., (2) with $\theta > 0$. When $T \to \infty$ and $\Delta_n \to 0$, we have*

$$D_{n,1} = O_p\left(e^{2\theta T}\right),\ D_{n,2} = O_p\left(e^{\theta T}\sqrt{T}\right),\ \textit{and}\ D_{3,n} = O_p(n^{1/2}).$$

Realized drift is dominated by the drift term, i.e.,

$$\mathsf{RD}_T = D_{n,1}\left[1 + o_p(1)\right] \sim_a \int_0^T \mu_s^2 ds = O_p\left(e^{2\theta T}\right)$$

if $e^{2\theta T}\Delta_n^{1/2}T^{-1/2} \to \infty$; and additionally if $e^{\theta T}\sqrt{\Delta_n} \to 0$ we have

$$\frac{1}{\sqrt{n}}\left[\mathsf{RD}_T - \int_0^T \mu_s^2 ds\right] \xrightarrow{d} \mathcal{MN}\left(0, \omega_{\sigma^4}\right). \tag{17}$$

In the Online Supplement it is shown that RD_T cannot reveal integrated drift when T is fixed under an explosive linear drift process. But as the time span T extends with $e^{2\theta T}\Delta_n^{1/2}T^{-1/2} \to \infty$, the drift term $D_{n,1}$ takes precedence and assumes a dominant role. Under double asymptotics and with the aforementioned conditions (viz., $e^{2\theta T}\Delta_n^{1/2}T^{-1/2} \to \infty$ and $e^{\theta T}\sqrt{\Delta_n} \to 0$), the limit property in (17) can be reformulated as follows:

$$\mathsf{RD}_T \sim_a \mathcal{MN}\left(\int_0^T \mu_s^2 ds,\ \Delta_n^{-1}\int_0^T \sigma_s^4 ds\right). \tag{18}$$

The statistic RD_T is an asymptotically unbiased estimator of ID_T. Nevertheless, like the standard model framework, its asymptotic variance diverges to infinity at the rate $O_p(nT)$.

Remark 3.1 *Consider the parameterization in which $T = -\kappa \log(\Delta_n)$, where $\kappa > 0$ is a constant. The condition*

$$\frac{e^{2\theta T}\Delta_n^{1/2}}{T^{1/2}} = \frac{\Delta_n^{1/2-2\kappa\theta}}{\sqrt{-\kappa\log(\Delta_n)}} \to \infty$$

implies that $\kappa\theta > \frac{1}{4}$, while the condition

$$e^{\theta T}\sqrt{\Delta_n} = \Delta_n^{1/2-\kappa\theta} \to 0$$

indicates that $\kappa\theta < \frac{1}{2}$. Here, θ represents the explosive rate of the linear drift and κ controls the duration of the explosive behavior. These conditions collectively suggest that $\kappa\theta$ must be greater than $1/4$ to detect the integrated drift, but it should remain less than $1/2$ to ensure that the variance of the estimator for RD_T behaves appropriately.

Asymptotics: Trending Drift

Under the trending linear drift process $\mu_s = \theta_0 + \theta_1 s$. Theorems 3.3 gives convergence properties of RD_T under double asymptotics.

Theorem 3.3 *Suppose p_t follows the trending drift process (8). When $T \to \infty$ and $\Delta_n \to 0$, we have*

$$D_{n,1} = O_p\left(T^3\right),\ D_{n,2} = O_p\left(\Delta_n T^{3/2}\right),\ D_{3,n} = O_p(n^{1/2}).$$

The realized drift estimator is dominated by the drift term such that

$$\mathsf{RD}_T = D_{n,1}\left[1 + o_p(1)\right] \sim_a \int_0^T \mu_s^2 ds = O_p\left(T^3\right)$$

if $T^{5/2}\Delta_n^{1/2} \to \infty$. Moreover, if, additionally, $T\Delta_n^{3/2} \to 0$, then we have

$$\frac{1}{\sqrt{n}}\left[\mathsf{RD}_T - \int_0^T \mu_s^2 ds\right] \xrightarrow{d} \mathcal{MN}\left(0, \omega_{\sigma^4}\right). \tag{19}$$

Again, in the Online Supplement it is shown that RD_T cannot reveal drift when T is fixed under a trending drift process. Theorem 3.3 shows that as the time span extends in such a way that $T^{5/2}\Delta_n^{1/2} \to \infty$, the drift term $D_{n,1}$ takes precedence, assuming a dominant role. The limit behavior (19) holds when $T^{5/2}\Delta_n^{1/2} \to \infty$ and $T\Delta_n^{3/2} \to 0$, and can be reformulated as

$$\mathsf{RD}_T \sim_a \mathcal{MN}\left(\int_0^T \mu_s^2 ds,\ \Delta_n^{-1}\int_0^T \sigma_s^4 ds\right), \tag{20}$$

suggesting that RD_T is an asymptotically unbiased estimator of ID_T but with a divergent asymptotic variance at the rate $O_p(nT)$.

Now assume that $T = c\Delta_n^{-\psi}$ with constant $c > 0$ and some parameter $\psi > 0$. The speed of divergence of T rises with the parameter ψ. We have $T\Delta_n \to 0$ when $\psi < 1$, $T\Delta_n = O_+(1)$ when $\psi = 1$, and $T\Delta_n \to \infty$ when $\psi > 1$. By definition, we have $n = T/\Delta_n = c\Delta_n^{-\psi-1} \to \infty$. The two required conditions can be rewritten as

$$\begin{aligned} T^{5/2}\Delta_n^{1/2} &= c\Delta_n^{-5/2\psi+1/2} \to \infty \Longrightarrow \psi > \frac{1}{5}, \\ T\Delta_n^{3/2} &= c\Delta_n^{-\psi+3/2} \to 0 \Longrightarrow \psi < \frac{3}{2}, \end{aligned}$$

which require that the time span T diverges to infinity faster than $\Delta_n^{-1/5}$ but slower than $\Delta_n^{-3/2}$.

Asymptotics: Drift Burst Process

Under the drift bursting process (9), the dynamics of asset returns at time t_i are given by

$$r_{t_i} = \int_{t_{i-1}}^{t_i} \mu_s (1 - \frac{s}{T})^{-\alpha} ds + \int_{t_{i-1}}^{t_i} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s.$$

The orders of the drift and diffusion components are, respectively,

$$\int_{t_{i-1}}^{t_i} \mu_s (1 - \frac{s}{T})^{-\alpha} ds = O_p\left(T^{\alpha} \Delta_n^{1-\alpha}\right) \text{ and } \int_{t_{i-1}}^{t_i} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s = O_p\left(T^{\beta} \Delta_n^{1/2-\beta}\right).$$

Under double asymptotics, volatility dominates the drift component when $n^{\beta-\alpha}\Delta_n^{-1/2} \to \infty$, and vice versa when $n^{\beta-\alpha}\Delta_n^{-1/2} \to 0$.

The realised drift estimator can be decomposed into three components such that

$$\mathsf{RD}_T = \tilde{D}_{1,n} + \tilde{D}_{2,n} + \tilde{D}_{3,n},$$

where

$$\begin{aligned}
\tilde{D}_{1,n} &= \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right), \\
\tilde{D}_{2,n} &= -\frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s \right) \\
&\quad -\frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right), \\
\tilde{D}_{3,n} &= \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} \sigma_s (1 - \frac{s}{T})^{-\beta} dW_s \right).
\end{aligned}$$

The limit properties of these quantities and RD_T are given in the next result.

Theorem 3.4 *Suppose that p_t follows the drift burst process specified in (9) with Assumption 2.1 and 2.2 and with $T \to \infty$ and $\Delta_n \to 0$. The asymptotic orders of the three terms are, respectively,*

$$\tilde{D}_{n,1} = O_p(T),\ \tilde{D}_{3,n} = O_p(T^{1/2}\Delta_n^{-1/2}),$$

and

$$\tilde{D}_{2,n} = \begin{cases} O_p\left(\sqrt{T}\right) & \text{if } \alpha + \beta < 1/2 \\ O_p\left(T^{1/2}\sqrt{\log n}\right) & \text{if } \alpha + \beta = 1/2 \\ O_p\left(T^{\alpha+\beta}\Delta_n^{1/2-\alpha-\beta}\right) & \text{if } \alpha + \beta > 1/2 \end{cases}.$$

The realized drift estimator is dominated by the drift term such that

$$\mathsf{RD}_T = \tilde{D}_{n,1}\left[1 + o_p(1)\right] \sim_a \int_0^T \mu_s^2 (1 - \frac{s}{T})^{-2\alpha} ds,$$

if $T\Delta_n \to \infty$. Let $\omega_{\sigma^4,\beta} \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \sigma_s^4 \left(1 - \frac{s}{T}\right)^{-4\beta} ds$. The limit distributions of RD_T is

$$n^{-1/2}\left[\mathsf{RD}_T - \int_0^T \mu_s^2 (1 - \frac{s}{T})^{-2\alpha} ds\right] \to_d \mathcal{N}\left(0, \omega_{\sigma^4,\beta}\right), \tag{21}$$

if the following conditions are satisfied:

$$\begin{cases} T\Delta_n \to \infty & \text{when } \alpha+\beta \le 1/2 \\ T\Delta_n \to \infty \text{ and } T^{1/2-(\alpha+\beta)}\Delta_n^{\alpha+\beta-1} \to \infty & \text{when } \alpha+\beta > 1/2 \end{cases}.$$

The drift bursting process behaves under Assumption 2.2 in a similar way asymptotically to the standard Ito semimartingale process. In particular, when $\alpha+\beta \le 1/2$, the condition required to obtain the limiting distribution is the same as that of the standard process in Theorem 3.1 (i.e., $T\Delta_n \to \infty$). When $\alpha+\beta > 1/2$, an additional condition (i.e., $T^{1/2-(\alpha+\beta)}\Delta_n^{-1+\alpha+\beta} \to \infty$) is required. Let $T = c\Delta_n^{-\psi}$ with $\psi > 0$. The two conditions imply that

$$1 < \psi < \gamma_{\alpha,\beta} \equiv \frac{1-(\alpha+\beta)}{\alpha+\beta-1/2}.$$

The time interval T diverges to infinity faster than Δ^{-1} but slower than $\Delta^{-\gamma_{\alpha,\beta}}$.

Remark 3.2 *Laurent et al. (2022) consider the data generating process (9) with $\alpha \in [0,1)$ and $\beta \in [0,1/2)$. They demonstrate that RD_T can only reveal drift under the infill setting when the rate of drift explosion is fast relative to volatility explosion, viz., when either (1) $\alpha > 3/4$, $\beta < 1/4$, and $\alpha+\beta < 1$, or (2) $\alpha+\beta < 1$ and $\alpha-\beta > 1/2$.*

4 Integrated Quarticity Estimators

Drawing from our earlier analysis and working within the framework of double asymptotics with careful consideration of parameter constraints, the realized drift estimator can effectively serve as an unbiased estimator for ID_T. Additionally, when normalized RD_T follows a mixture normal distribution with its conditional variance equaling the integrated quarticity IQ_T. For the purpose of conducting statistical inference on the drift, it is therefore imperative to obtain a consistent estimator for IQ_T under the various model specifications.

We consider two quarticity estimators: realised quarticity RQ_T and RiceQ_T. The realised quarticity estimator (Barndorff-Nielsen and Shephard, 2003) is defined as

$$\mathsf{RQ}_T = \frac{1}{3\Delta_n}\sum_{i=1}^{n}\left(r_{n,i}\right)^4.$$

An alternative estimator, RiceQ_T, was introduced by Laurent et al. (2022) with the intention of mitigating the impact of drift and jumps. It is constructed using the following formula:

$$\mathsf{RiceQ}_T = \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(r_{n,i}-r_{n,i-1}\right)^2\left(r_{n,i-1}-r_{n,i-2}\right)^2.$$

We begin by examining the asymptotic characteristics of RQ_T and RiceQ_T under the standard model specification (1), where the infill asymptotics of RQ_T are well known. Specifically, under Assumption 2.1, when $\Delta_n \to 0$ and T is held constant, RQ_T consistently estimates integrated quarticity, denoted by IQ_T:

$$\mathsf{RQ}_T \to_p \mathsf{IQ}_T.$$

However, the behavior of RQ_T under double asymptotics has not yet been investigated. The consistency of RiceQ_T was investigated in LRS under the infill asymptotic setting but the performance of RiceQ_T when $T \to \infty$ and $\Delta_n \to 0$ is yet to be explored. The following result delivers double asymptotics in both cases.

Theorem 4.1 *Suppose the log price p_t follows the standard Ito semimartingale process (1) with Assumption 2.1. As $T \to \infty$ and $\Delta_n \to 0$,*

$$\mathsf{RQ}_T = \int_0^T \sigma_s^4 ds + O_p\left(T\Delta_n\right) \text{ and } \mathsf{RiceQ}_T = \int_0^T \sigma_s^4 ds + O_p\left(T\Delta_n^{1+\Gamma/2}\right).$$

Theorem 4.1 shows that both RQ_T and RiceQ_T are asymptotically dominated by IQ_T under double asymptotics. Moreover, the RiceQ_T estimator, which is obtained from $\Delta r_{n,i} \equiv r_{n,i} - r_{n,i-1}$, has a smaller bias than RQ_T which is computed from $r_{n,i}$. Specifically, the biased term of RiceQ_T is $O_p(T\Delta_n^{1+\Gamma/2})$ with $\Gamma > 0$, as opposed to $O_p(T\Delta_n)$ for RQ_T.

To provide some intuition for this finding we examine the asymptotic orders of the drift and diffusion terms for $r_{n,i}$ and $\Delta r_{n,i}$. Under the data generating process, the return process is defined as follows:

$$r_{n,i} = \int_{t_{i-1}}^{t_i} \mu_s ds + \int_{t_{i-1}}^{t_i} \sigma_s dW_s.$$

Since μ and σ are locally bounded, $\int_{t_{i-1}}^{t_i} \mu_s ds = O_p\left(\Delta_n\right)$ and $\int_{t_{i-1}}^{t_i} \sigma_s dW_s = O_p\left(\Delta_n^{1/2}\right)$. The first differenced return is given by:

$$\Delta r_{n,i} = \underbrace{\int_{t_{i-1}}^{t_i} \left(\mu_s - \mu_{s-\Delta_n}\right) ds}_{\text{drift}} + \underbrace{\left[\int_{t_{i-1}}^{t_i} \sigma_s dW_s - \int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s\right]}_{\text{diffusion}}.$$

The diffusion term of $\Delta r_{n,i}$ is a zero mean Gaussian process with variance $2\sigma_{t_{i-1}}^2 \Delta_n \left[1 + o_p\left(1\right)\right]$, implying that the diffusion term is $O_p\left(\Delta_n^{1/2}\right)$. Under the Hölder continuity assumption,

$$\left|\int_{t_{i-1}}^{t_i} \left(\mu_s - \mu_{s-\Delta_n}\right) ds\right| \leq \int_{t_{i-1}}^{t_i} \left|\mu_s - \mu_{s-\Delta_n}\right| ds \leq K\Delta_n^{1+\Gamma/2}.$$

While both $r_{n,i}$ and $\Delta r_{n,i}$ share the same asymptotic order for their diffusion terms, it is noteworthy that the drift term of $\Delta r_{n,i}$ has a comparatively smaller order of magnitude, namely $O_p\left(\Delta_n^{1+\Gamma/2}\right)$, in contrast to the $O_p\left(\Delta_n\right)$ order for $r_{n,i}$. This distinction is significant, as demonstrated in Theorem 4.1 where RiceQ_T was shown to have smaller bias than RQ_T. Consequently, our primary attention is focused on exploring the asymptotic properties of RiceQ_T for other drift processes.

The following results establish consistency of RiceQ_T under explosive linear drift, trending drift, and bursting drift processes in the double asymptotic framework.[7]

Theorem 4.2 *Suppose the log price p_t follows the explosive linear drift mechanism in model (2) with $\theta > 0$. When $T \to \infty$, $\Delta_n \to 0$, and $e^{\theta T}\sqrt{\Delta_n} \to 0$,*

$$\mathsf{RiceQ}_T = \int_0^T \sigma_s^4 ds + O_p\left(e^{2\theta T}\Delta_n^2\right).$$

Hence, under explosive linear drift the RiceQ_T statistic is a consistent estimator for IQ_T under a double asymptotic scheme where $T \to \infty$ at a relatively slow rate, specifically when $e^{\theta T}\sqrt{\Delta_n} \to 0$. The same condition is required for RD_T in (17).

[7] In the Online Supplement it is shown that RiceQ_T consistently estimates IQ_T across all model specifications under infill asymptotics.

Theorem 4.3 *Suppose the log price p_t follows the trending drift process (8). Under double asymptotics when $T\Delta_n \to 0$,*

$$\mathsf{RiceQ}_T = \int_0^T \sigma_s^4 ds + O_p\left(T^2\Delta_n\right).$$

Thus, when log prices are generated from a trending drift process, RiceQ_T again consistently estimates IQ_T under double asymptotics; and, similar to the explosive linear drift process, consistency requires that the rate of divergence of T cannot be too fast. The requirement on T is that $T\Delta_n \to 0$ or $T = c\Delta_n^{-\psi}$ with $\psi < 1$.

Theorem 4.4 *Suppose that p_t follows the drift bursting process specified in (9) under Assumptions 2.1 and 2.2. Under double asymptotics,*

$$\mathsf{RiceQ}_T = \int_0^T \sigma_s^4 (1-s)^{-4\beta} ds + O_p\left(\max\left\{T^{4\alpha}\Delta_n^{3-4\alpha}, T^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}\right\}\right)$$

when $\alpha \leq 1/4$. But when $\alpha > 1/4$ the following additional conditions are required:

$$\begin{cases} T^{1-4\alpha}\Delta_n^{4\alpha-3} \to \infty & \text{if } \alpha+\beta \leq 1/2 \\ T^{1-4\alpha}\Delta_n^{4\alpha-3} \to \infty \text{ and } T^{1/2-(\alpha+\beta)}\Delta_n^{\alpha+\beta-1} \to \infty & \text{if } \alpha+\beta > 1/2 \end{cases}.$$

Under drift bursting when T is fixed and $\Delta_n \to 0$, RiceQ_T is consistent for IQ_T, as shown in the Online Supplement. Also when $T \to \infty$ if the rate of drift explosion is slow (i.e., $\alpha \leq 1/4$), consistency of RiceQ_T remains intact. Further, if the rate of drift bursting exceeds certain thresholds (viz., $\alpha > 1/4$) then similar to explosive linear drift and trend drift processes, the rate at which T diverges must be relatively slow to ensure the consistency of RQ_T. Again, let $T = c\Delta_n^{-\psi}$. The condition implies that

$$\begin{cases} \psi < \frac{3-4\alpha}{4\alpha-1} & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta \leq 1/2 \\ \psi < \min\left\{\frac{3-4\alpha}{4\alpha-1}, \frac{1-(\alpha+\beta)}{\alpha+\beta-1/2}\right\} & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta > 1/2 \end{cases}.$$

In both cases, the upper bound approaches unity when $\alpha \to 1/2$ (which is the worse possible scenario).

5 Statistical Inference

Testing for nonzero integrated drift is important in financial analysis and practice as it helps investors identify potential long-term trends and make informed investment decisions, while also enabling risk assessment and accurate modeling for reliable forecasting. Detecting nonzero drift can significantly impact both individual and institutional financial strategies. For this purpose, we consider the following test statistic

$$S_T = \frac{|\,\mathsf{RD}_T\,|}{\sqrt{\mathsf{RiceQ}_T\,/\Delta_n}}. \tag{22}$$

Corollary 5.1 *The conditions for ensuring consistency of the drift test across the four drift diffusion processes are as below. Under the specified conditions, the test statistic S_T has a standard normal limit distribution*

$$S_T \xrightarrow{d} Z \equiv \mathcal{N}(0,1)$$

under the null hypothesis of zero drift. Under the alternative hypothesis, as $T \to \infty$ and $\Delta_n \to 0$, S_T diverges at varying rates as given below in (i)–(iv):

$$S_T = \frac{|\,\mathsf{RD}_T\,|}{\sqrt{\mathsf{RiceQ}_T\,/\Delta_n}} \sim_a Z + \frac{\int_0^T \mu_s^2 ds}{\sqrt{\mathsf{ID}_T\,/\Delta_n}} \to \infty.$$

(i) ***Standard Ito semimartingale process:*** $T\Delta_n \to \infty$.

(ii) ***Explosive linear drift process:***

$$e^{2\theta T}\Delta_n^{1/2}T^{-1/2} \to \infty \text{ and } e^{\theta T}\sqrt{\Delta_n} \to 0.$$

When we let $T = \log(\Delta_n^{-\kappa})$ *with* $\kappa > 0$*, these conditions imply* $\frac{1}{4} < \kappa\theta < \frac{1}{2}$.

(iii) ***Trending drift process:***

$$T^{5/2}\Delta_n^{1/2} \to \infty \text{ and } T\Delta_n \to 0.$$

Letting $T = c\Delta_n^{-\psi}$ *with* $c > 0$ *and* $\psi > 0$*, these conditions imply* $\frac{1}{5} < \psi < 1$.

(iv) ***Drift burst process:***

$$\begin{cases} T\Delta_n \to \infty & \text{if } \alpha \le 1/4 \\ T\Delta_n \to \infty \text{ and } T^{1-4\alpha}\Delta_n^{4\alpha-3} \to \infty & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta \le 1/2 \\ T\Delta_n \to \infty \text{ and } T^{1/2-(\alpha+\beta)}\Delta_n^{\alpha+\beta-1} \to \infty \text{ and } T^{1-4\alpha}\Delta_n^{4\alpha-3} \to \infty & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta > 1/2 \end{cases}$$

Again, letting $T = c\Delta_n^{-\psi}$*, the conditions can be translated to the following:*

$$\begin{cases} \psi > 1 & \text{if } \alpha \le 1/4 \\ 1 < \psi < \frac{3-4\alpha}{4\alpha-1} & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta \le 1/2 \\ 1 < \psi < \min\left\{\frac{3-4\alpha}{4\alpha-1}, \frac{1-(\alpha+\beta)}{\alpha+\beta-1/2}\right\} & \text{if } \alpha > 1/4 \text{ and } \alpha+\beta > 1/2 \end{cases}.$$

The proof of Corollary 5.1 follows directly from the preceding results. If z_s is the s percentile of the standard normal, then the null hypothesis of zero drift is rejected when the test statistic is greater than $z_{1-\alpha/2}$ at the $\alpha\%$ level. The confidence interval for ID_T can be constructed in the usual way as

$$\left[\mathsf{RD}_T - \sqrt{\frac{\mathsf{RiceQ}_T}{\Delta_n}}z_{1-\alpha/2}, \mathsf{RD}_T - \sqrt{\frac{\mathsf{RiceQ}_T}{\Delta_n}}z_{\alpha/2}\right],$$

such that the probability

$$\Pr\left\{\mathsf{RD}_T - \sqrt{\frac{\mathsf{RiceQ}_T}{\Delta_n}}z_{1-\alpha/2} \le \mathsf{ID}_T \le \mathsf{RD}_T - \sqrt{\frac{\mathsf{RiceQ}_T}{\Delta_n}}z_{\alpha/2}\right\} = 1-\alpha$$

under the stated conditions for each data generating process.

6 Simulations

The simulation study has two aims: (i) to examine the accuracy in finite samples of RD as an estimate of integrated drift; and (ii) to investigate the size and power of two zero drift tests, where the test statistics are defined as

$$S_{1,T} = \frac{|\,\mathsf{RD}_T\,|}{\sqrt{\mathsf{RQ}_T/\Delta_n}} \text{ and } S_{2,T} = \frac{|\,\mathsf{RD}_T\,|}{\sqrt{\mathsf{RiceQ}_T/\Delta_n}}. \tag{23}$$

The null hypothesis of zero drift is rejected when the test statistic is greater than $z_{1-\alpha/2}$ at the $\alpha\%$ level. The test based on RiceQ (i.e., $S_{2,T}$) is expected to have superior power as it provides more accurate estimation of IQ according to the limit theory in Section 5.[8]

[8] We discuss jump-robust versions of $S_{1,T}$ and $S_{2,T}$ and examine their performance in the presence of jumps in the Appendix.

6.1 Data generating processes

Without loss of generality, we assume there is no volatility explosion (i.e., $\beta = 0$) and allow the drift term to take four forms: constant drift, linear drift, trending drift, and drift burst, as follows:

$$\begin{array}{ll} \text{Constant drift:} & \mathrm{d}\,p_t = \mu\,\mathrm{d}\,t + \sigma_t\,\mathrm{d}\,W_t \\ \text{Linear drift:} & \mathrm{d}\,p_t = \theta p_t\,\mathrm{d}\,t + \sigma_t\,\mathrm{d}\,W_t \\ \text{Trending drift:} & \mathrm{d}\,p_t = \theta_1 t\,\mathrm{d}\,t + \sigma_t\,\mathrm{d}\,W_t \\ \text{Drift burst:} & \mathrm{d}\,p_t = \mu(1-t/T)^{-\alpha}\,\mathrm{d}\,t + \sigma_t\,\mathrm{d}\,W_t \end{array}.$$

The first two specifications of nonzero drift have been employed frequently in the literature,[9] while the drift burst process was employed in Laurent et al. (2022). The diffusion coefficient σ_t follows the commonly used Heston (1993) type stochastic volatility model, as in Zhang et al. (2005); Bandi and Russell (2006); Ait-Sahalia et al. (2007); Christensen et al. (2020). Specifically,

$$\mathrm{d}\,\sigma_t^2 = \kappa\left(\omega - \sigma_t^2\right)\mathrm{d}\,t + \gamma\sigma_t\,\mathrm{d}\,W_t^\sigma, \tag{24}$$

where W_t^σ is standard Brownian motions with cross covariance $E\left(\mathrm{d}\,W_t\,\mathrm{d}\,W_t^\sigma\right) = \rho\,\mathrm{d}\,t$ with W_t. The annual parameters of the model are $(\kappa, \omega, \gamma, \rho) = (5, 0.0225, 0.4, -\sqrt{0.5})$ as in Christensen et al. (2020). The value of ω implies an unconditional standard deviation of log returns of roughly 15% per annum. In each simulation, σ_t^2 is initiated at random from its stationary two parameter gamma distribution law $\sigma_t^2 \sim \Gamma\left(2\kappa\omega\gamma^{-2}, 2\kappa\gamma^{-2}\right)$.

The noise-contaminated log price is denoted by p_t^o and defined as

$$p_t^o = p_t + v_t z_t \text{ with } z_t \sim_{i.i.d} \mathcal{N}(0,1), \tag{25}$$

where $v_t = \sqrt{\Delta_n}\xi\sigma_t$ and ξ is the noise-to-signal ratio. The noise is conditionally heteroskedastic, serial dependent (via σ_t), and positively related to the riskiness of the efficient log-price. The parameter $\xi = 0.5$, which amounts to a median contamination level. See Aït-Sahalia et al. (2012); Christensen et al. (2014, 2019). [10]

We simulate data series over a period of T days at the one second frequency with 6.5 trading hours each day (i.e., $\mathrm{d}\,t = 1/23,400$). The opening price each day is affected by overnight information. To further our understanding of the impact, we examine the Nasdaq composite index employed in our application section. Figure 1 provides two examples of different types of overnight price jumps. Evidently, overnight price jumps can be of very different sizes.

Denote the log opening and closing (noisy) prices on the ith day by $p_{t_i}^o$ and $p_{t_i+n^*\Delta_n}^o$, respectively, where n^* is the number of one-second returns within each day. The daily return on day t_i is defined as the closing price less the opening price (i.e., $r_{t_i}^o = p_{t_i+n^*\Delta_n}^o - p_{t_i}^o$) and the overnight return is the difference between the opening price of day t_i and the closing price of day t_{i-1} (i.e., $\tilde{r}_{t_i}^o = p_{t_i}^o - p_{t_{i-1}+n^*\Delta_n}^o$). In Figure 2(a), we plot the ratio between overnight and daily returns

$$v_{t_i} = \tilde{r}_{t_{i+1}}^o / r_{t_i}^o$$

against the daily return. The two vertical lines in the graph indicate locations of $r_{t_i}^o = -0.003$ and $r_{t_i}^o = 0.003$. The ratio is evidently extremely large when the return is very close to zero (i.e., $|r_{t_i}^o| \leq 0.003$). Let $v_t^* = v_t \mathbf{1}(|r_{t_i}^o| > 0.003)$, where $\mathbf{1}(.)$ is the indicator function. Figure 2(b) plots the kernel density of the ratio v_t^*. As evident in the graph, after removing extreme values the ratio is centred at unity and leptokurtic. Consequently, in our simulations the overnight return will be generated as $\tilde{r}_{t_{i+1}}^o = r_{t_i}^o \tilde{v}_{t_i}^*$ with $\tilde{v}_{t_i}^*$ being a random draw from the $v_{t_i}^*$ sequence of the Nasdaq composite index.

Figure 1: The log prices of the Nasdaq composite index at the one-second frequency. The vertical line indicates the market opening of the second day.

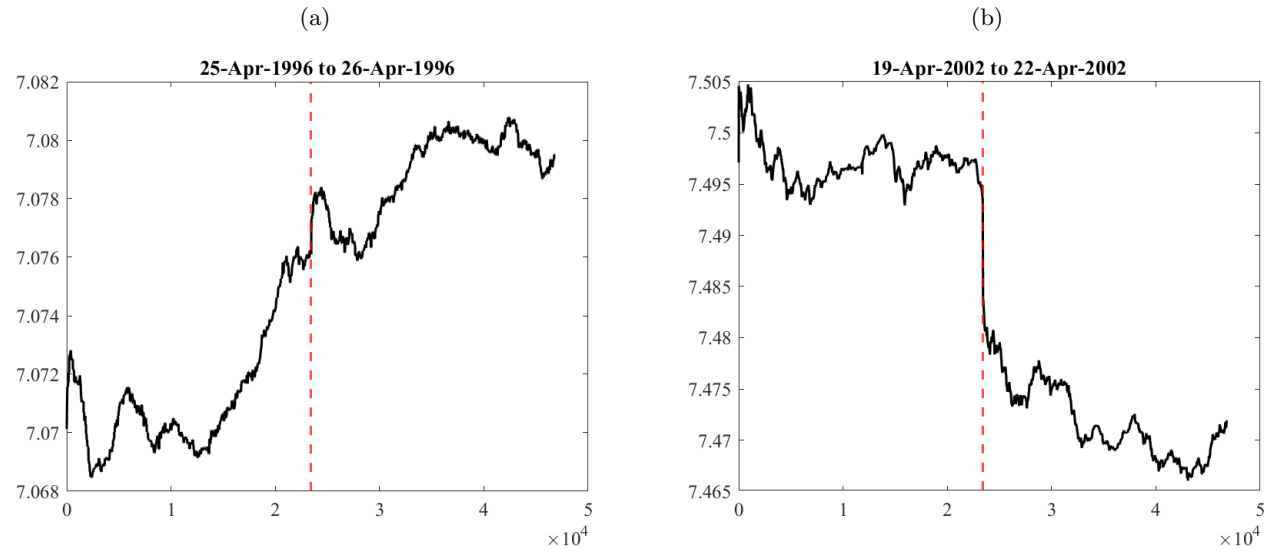


Figure 2: The ratio between overnight and daily returns: the Nasdaq Composite Index 1996 to 2020

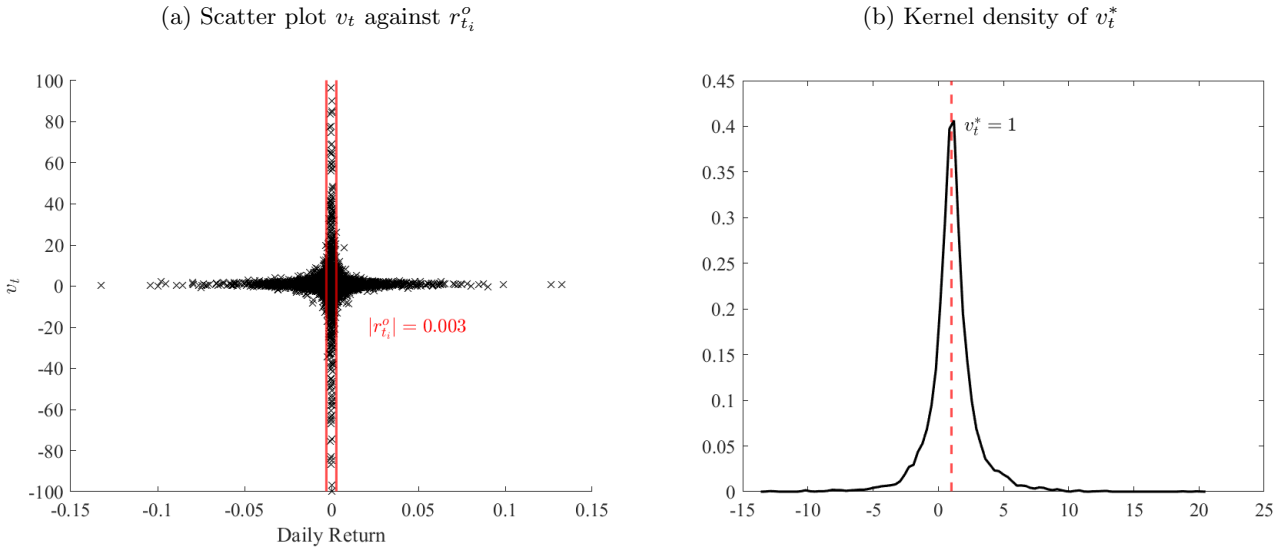


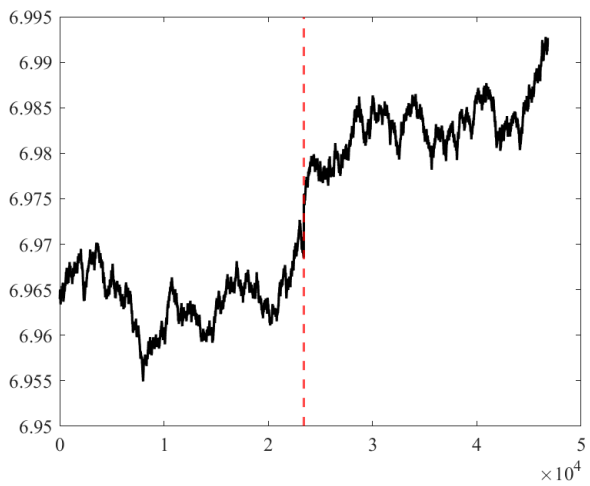
Figure 3 shows four typical sample paths of the data generating process at the one-second frequency with $T = 2$ and $p_0 = 6.96$ (which is the first value of the Nasdaq series), one for each of the four DGPs. The first graph is from the constant drift specification with $\mu = 0.01$; the second is from the line drift model with $\theta = 0.001$; the third is from the trending drift process with $\theta_1 = -0.02$; and the last is from the explosive drift specification with $\mu = 0.02$ and $\alpha = 0.2$. The simulated data series resemble those of

[9]See, e.g., Lo and Wang (1995); Barndorff-Nielsen and Shephard (2001); Nicolato and Venardos (2003); Aalen and Gjessing (2004); Zhou and Yu (2015); Wang and Yu (2016).

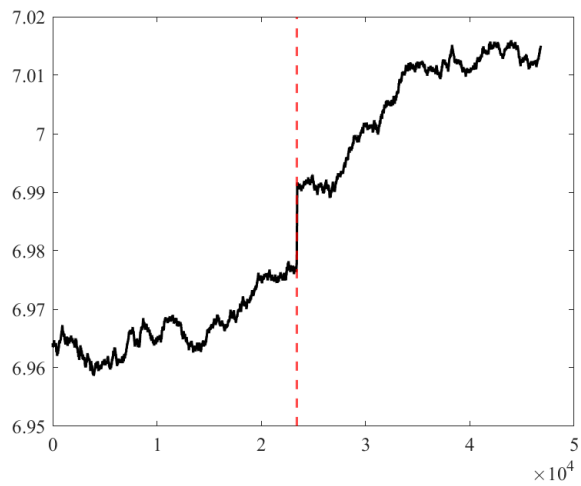
[10]In the Appendix, we consider a noise contaminated jump diffusion process and analyze how jumps affect both the realized drift estimator and the zero drift test.

Figure 3: Typical sample paths of the data generating process at the one-second frequency with $T = 2$. The vertical line indicates the opening of the second day.

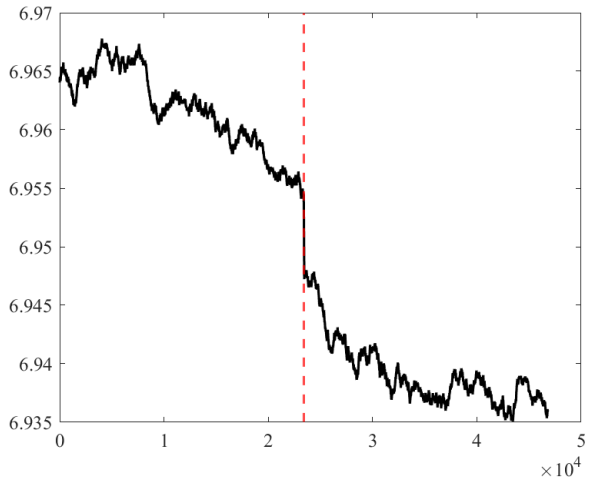
(a) Constant Drift: $\mu = 0.01$



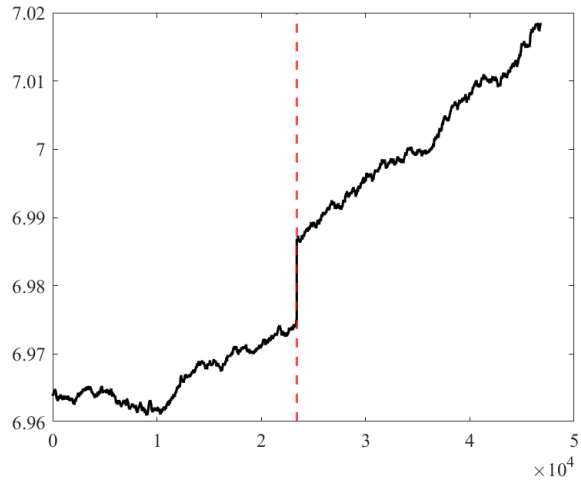
(b) Explosive Linear Drift: $\theta = 0.001$



(c) Trending Drift: $\theta_1 = -0.02$



(d) Bursting Drift: $\mu = 0.02$ and $\alpha = 0.2$



the Nasdaq stock market presented in Figure 1.

6.2 Results

For all subsequent simulations, we simulate data at the one-second frequency and aggregate returns over non-overlapping 5-minute windows to minimise the impact of market microstructure noise, as is standard in the literature (e.g., Zhang et al. (2005)). We exclude the overnight returns and the first 5-minute return when computing RD_T, RQ_T, and RiceQ_T. The simulation is repeated 2,000 times. For the constant drift setting, the drift coefficient μ varies from 0.01 to 0.05 with increments of 0.02. In the case of the linear drift specification, we allow the parameter θ to increase from -0.005 to 0.005 with increments of 0.002. The θ_1 parameter of the trending drift process varies from -0.01 to -0.05 with increments of 0.02. For the drift burst specification, we fix μ at 0.02 and allow α to increase from 0.1 to 0.3 with a step size

of 0.1. The time period T is either one day, one week, two weeks, or one month (i.e., $T = 1, 5, 10, 20$, respectively).

Table 1 shows the accuracy of RD_T as an estimate of integrated drift ID_T, reporting means and standard deviations of $\frac{1}{T}(\mathsf{RD}_T - \mathsf{ID}_T) \times 10^3$. Evidently, the bias magnitudes are small under all settings. As the sample period increases from a day to a month, the standard deviation reduces by approximately 75% whereas reduction in bias is much less apparent. This finding holds for all drift scenarios with different parameter values. As anticipated, the bias and standard deviation (in absolute terms) are both greater in magnitude when the drift is larger.

Table 1: The mean and standard deviation (in parentheses) of $\frac{1}{T}(\mathsf{RD} - \mathsf{ID}) \times 10^3$ under various parameter settings. The number of replications is $2,000$.

Coeff	$T = 1$	$T = 5$	$T = 10$	$T = 20$
Zero Drift				
$\mu = 0$	-0.01 (0.91)	0.01 (0.43)	0.01 (0.31)	0.00 (0.22)
Constant Drift				
$\mu = 0.01$	-0.00 (0.98)	0.02 (0.46)	-0.01 (0.30)	-0.01 (0.23)
$\mu = 0.03$	-0.06 (1.15)	-0.03 (0.49)	-0.01 (0.33)	-0.03 (0.25)
$\mu = 0.05$	-0.13 (1.30)	-0.09 (0.58)	-0.08 (0.41)	-0.09 (0.29)
Stationary Linear Drift				
$\theta = -0.001$	-0.00 (1.03)	0.00 (0.43)	-0.00 (0.32)	-0.00 (0.22)
$\theta = -0.003$	0.01 (1.08)	-0.03 (0.46)	-0.01 (0.33)	-0.01 (0.24)
$\theta = -0.005$	-0.02 (1.12)	-0.04 (0.50)	-0.04 (0.36)	-0.04 (0.24)
Explosive Linear Drift				
$\theta = 0.001$	0.00 (0.96)	0.01 (0.46)	0.00 (0.31)	-0.01 (0.23)
$\theta = 0.003$	-0.06 (1.03)	-0.01 (0.48)	-0.02 (0.32)	-0.02 (0.24)
$\theta = 0.005$	-0.04 (1.22)	-0.08 (0.54)	-0.05 (0.35)	-0.06 (0.26)
Trending Drift				
$\theta_1 = -0.01$	-0.01 (0.99)	-0.01 (0.43)	-0.00 (0.31)	-0.00 (0.22)
$\theta_1 = -0.03$	-0.02 (1.05)	-0.00 (0.47)	-0.01 (0.33)	-0.01 (0.24)
$\theta_1 = -0.05$	-0.07 (1.09)	-0.03 (0.49)	-0.04 (0.35)	-0.03 (0.26)
Bursting Drift: $\mu = 0.02$				
$\alpha = 0.1$	-0.02 (1.08)	-0.02 (0.48)	-0.01 (0.34)	-0.02 (0.23)
$\alpha = 0.2$	-0.07 (1.10)	-0.03 (0.46)	-0.04 (0.35)	-0.03 (0.24)
$\alpha = 0.3$	-0.16 (1.11)	-0.13 (0.48)	-0.10 (0.34)	-0.10 (0.26)

Table 2 reports size and power of the zero drift tests based on $S_{1,T}$ and $S_{2,T}$ with nominal size of 10%. Both tests display strong performance in all finite sample scenarios, with size near nominal across all T settings. Test power increases as span expands from one day to one month. For instance, under the constant drift setup, with a drift coefficient $\mu = 0.03$, the power of the $S_{2,T}$ test rises from 53% to 94% when T increases from unity to 20. Similar improvements occur for the test based on the $S_{1,T}$ statistic and when the DGP is either a linear, trending, or bursting drift process. These findings corroborate the limit theory of Section 3 which established that the zero drift test is consistent under certain restrictions on T. As expected, the tests have higher detective capability for non-zero drift as the magnitude of the drift component rises. With the explosive linear drift DGP and $T = 1$, the powers of the S_1 test rises from 12% to 57% when $|\theta|$ grows from 0.001 to 0.005. The same trend is visible for the drift burst process when the explosive rate α rises from 0.1 to 0.3. Finally, it is worth noting that the power of S_2 consistently surpasses or matches that of $S_{1,T}$. This result aligns with our analysis in Section 5, where RiceQ was

shown to deliver a more precise estimate than RQ in the presence of drift, enabling more accurate inference concerning drift. The power advantage of $S_{2,T}$ over $S_{1,T}$ is substantial in some cases. For instance, when $T = 1$ and $\alpha = 0.3$ under the bursting drift scenario, the power of $S_{2,T}$ is 8% higher than that of S_1.

Table 2: Sizes and powers of the drift tests under various parameter settings. The nominal size of the tests is 10%. The number of replications is $2,000$.

Coeff	$T=1$		$T=5$		$T=10$		$T=20$	
	$S_{1,T}$	$S_{2,T}$	$S_{1,T}$	$S_{2,T}$	$S_{1,T}$	$S_{2,T}$	$S_{1,T}$	$S_{2,T}$
Zero Drift								
$\mu = 0$	0.11	0.11	0.10	0.11	0.09	0.10	0.10	0.11
Constant Drift								
$\mu = 0.01$	0.15	0.19	0.23	0.26	0.24	0.27	0.31	0.33
$\mu = 0.03$	0.46	0.53	0.74	0.76	0.87	0.89	0.93	0.94
$\mu = 0.05$	0.76	0.81	0.96	0.97	0.99	0.99	1.00	1.00
Stationary Linear Drift								
$\theta = -0.001$	0.12	0.15	0.13	0.15	0.14	0.16	0.13	0.15
$\theta = -0.003$	0.31	0.38	0.49	0.54	0.60	0.63	0.71	0.73
$\theta = -0.005$	0.54	0.63	0.83	0.86	0.90	0.91	0.96	0.97
Explosive Linear Drift								
$\theta = 0.001$	0.12	0.15	0.14	0.17	0.16	0.18	0.18	0.20
$\theta = 0.003$	0.31	0.37	0.56	0.59	0.66	0.67	0.77	0.78
$\theta = 0.005$	0.57	0.63	0.82	0.85	0.93	0.93	0.98	0.98
Trending Drift								
$\theta_1 = -0.01$	0.11	0.14	0.12	0.13	0.11	0.12	0.13	0.14
$\theta_1 = -0.03$	0.23	0.29	0.42	0.45	0.50	0.53	0.63	0.65
$\theta_1 = -0.05$	0.45	0.52	0.72	0.76	0.84	0.85	0.92	0.93
Bursting Drift: $\mu = 0.02$								
$\alpha = 0.1$	0.33	0.40	0.55	0.58	0.67	0.69	0.78	0.79
$\alpha = 0.2$	0.38	0.45	0.65	0.69	0.75	0.77	0.87	0.88
$\alpha = 0.3$	0.45	0.53	0.75	0.79	0.85	0.86	0.94	0.94

7 Empirical Applications: Nasdaq Stock Market

The Nasdaq composite index was downloaded at the one-second frequency from Refinitive Tick History over January 2, 1996 to December 11, 2020 and observations outside normal trading hours (9:30 to 16:00) were removed. We cleaned the data according to standard practice (Barndorff-Nielsen et al., 2009; Boudt et al., 2021).

Figure 4 displays the evolution of the data series. The Nasdaq composite index experienced significant volatility and growth. The index started the period around $1,300$ and reached a value of over $9,000$ in February 2020, before dropping sharply in March of that year due to the COVID-19 pandemic. It recovered quickly in April 2020 and reached a new high of over $12,000$ at the end of the sample period. There were several significant events that impacted the index during this period. For example, the dot-com bubble of the late 1990s and early 2000s caused the index to reach a high of over 5,000 in March 2000 before crashing and losing more than 75% of its value over the following two years. A subsequent significant correction occurred in 2016-2017. The index then resumed its upward trend, driven by strong earnings from technology companies and low interest rates. Overall, the Nasdaq Composite Index has been one

of the best-performing stock market indices over the past 25 years, reflecting the growth and dominance of the technology sector in the global economy. But its performance has also been subject to significant volatility and market corrections.

Figure 4: Nasdaq composite index: from January 2, 1996 to December 11, 2020

High-frequency financial data is often affected by microstructure noise which can have a significant impact on analysis results. To address this issue a popular solution is to use returns computed from sparsely sampled prices (i.e., aggregated returns over non-overlapping windows); see, e.g., Zhang et al. (2005) for details. We used the popular choice of five-minute frequency for the Nasdaq composite index.

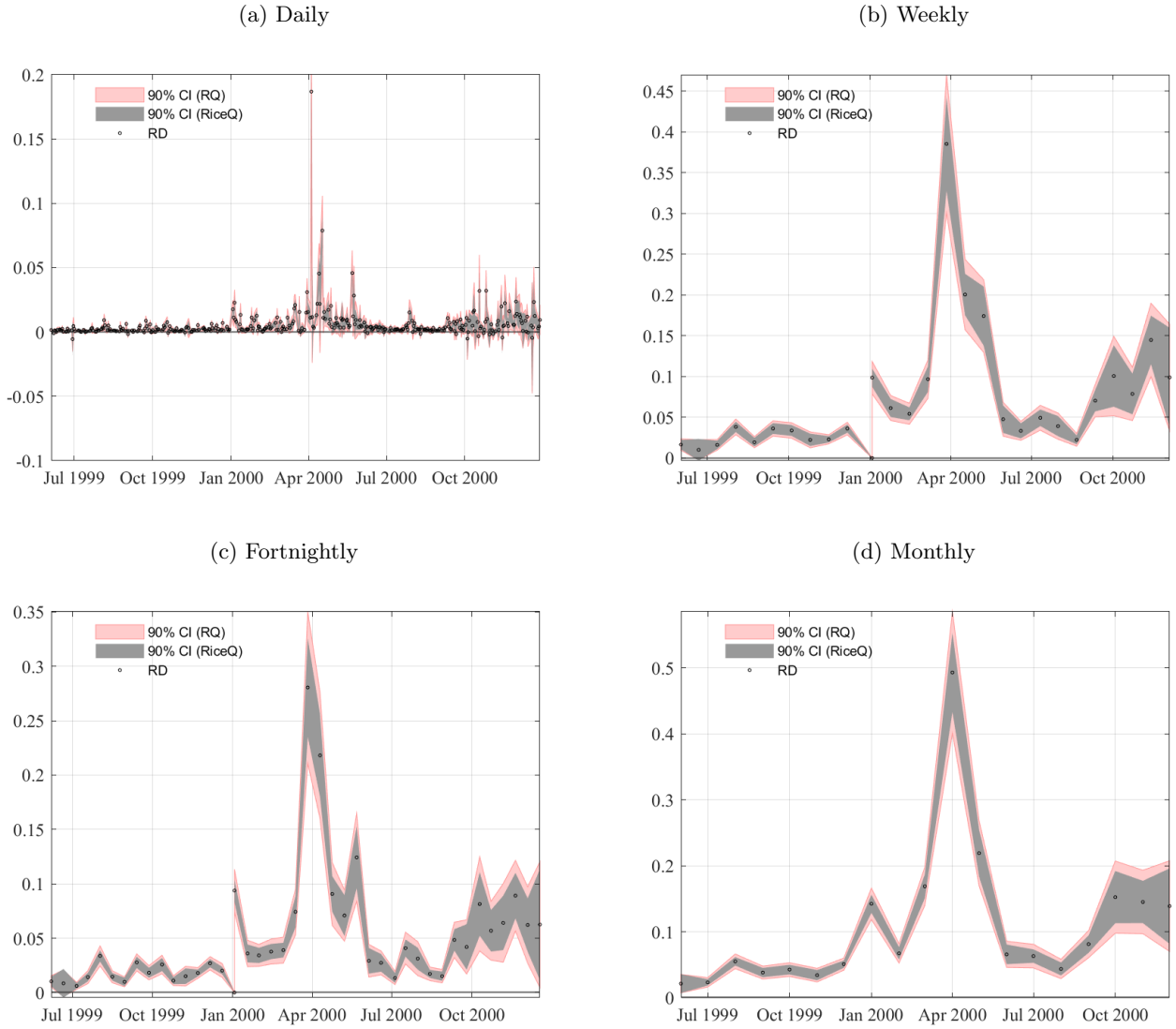
We computed the daily, weekly, fortnightly, and monthly realized drifts of the Nasdaq composite index along with their corresponding confidence intervals based on RiceQ. Overnight returns and first returns of the day were excluded when computing RD, RQ, and RiceQ. To highlight the importance of using the RiceQ estimator instead of RQ for making inferences Figure 5 shows confidence bands constructed from both IQ estimators for the period between June 1999 and December 2000. As evident in the figure the confidence bands based on RiceQ are narrower than those from RQ, resulting in sharper inference. This gap between the two confidence bands is notably more substantial during volatile periods; but as the time span T increases from one day to one month the difference between the two confidence bands reduces.

Figure 6 presents results over the entire period displaying the RD estimator and its 90% confidence interval; Figure 7 displays only the significant drift estimates. The graphics reveal a significant deviation from zero drift during the dot-com bubble expansion and the bursting periods from 1996 to 2001, with the drift value reaching its peak in April 2000. This result is consistent with previous studies that used either intraday high-frequency data (Laurent et al., 2022; Laurent and Shi, 2020) or lower frequency data (Phillips et al., 2011; Shi and Song, 2016). Some small but significant deviations from zero drift were also identified between 2004 and 2007; and the analysis shows that the subprime mortgage crisis led to a moderate sized drift deviation in the Nasdaq price index in April 2009.

8 Conclusion

Various economic events, including financial bubbles, crises, and flash crises, have the potential to induce drift deviations in asset prices. Price dynamics during such episodes can be characterized by distinct model specifications. This paper explores the possibility of detecting drift deviations using a realized drift measure constructed from intraday high-frequency data within a certain time span T. Four different

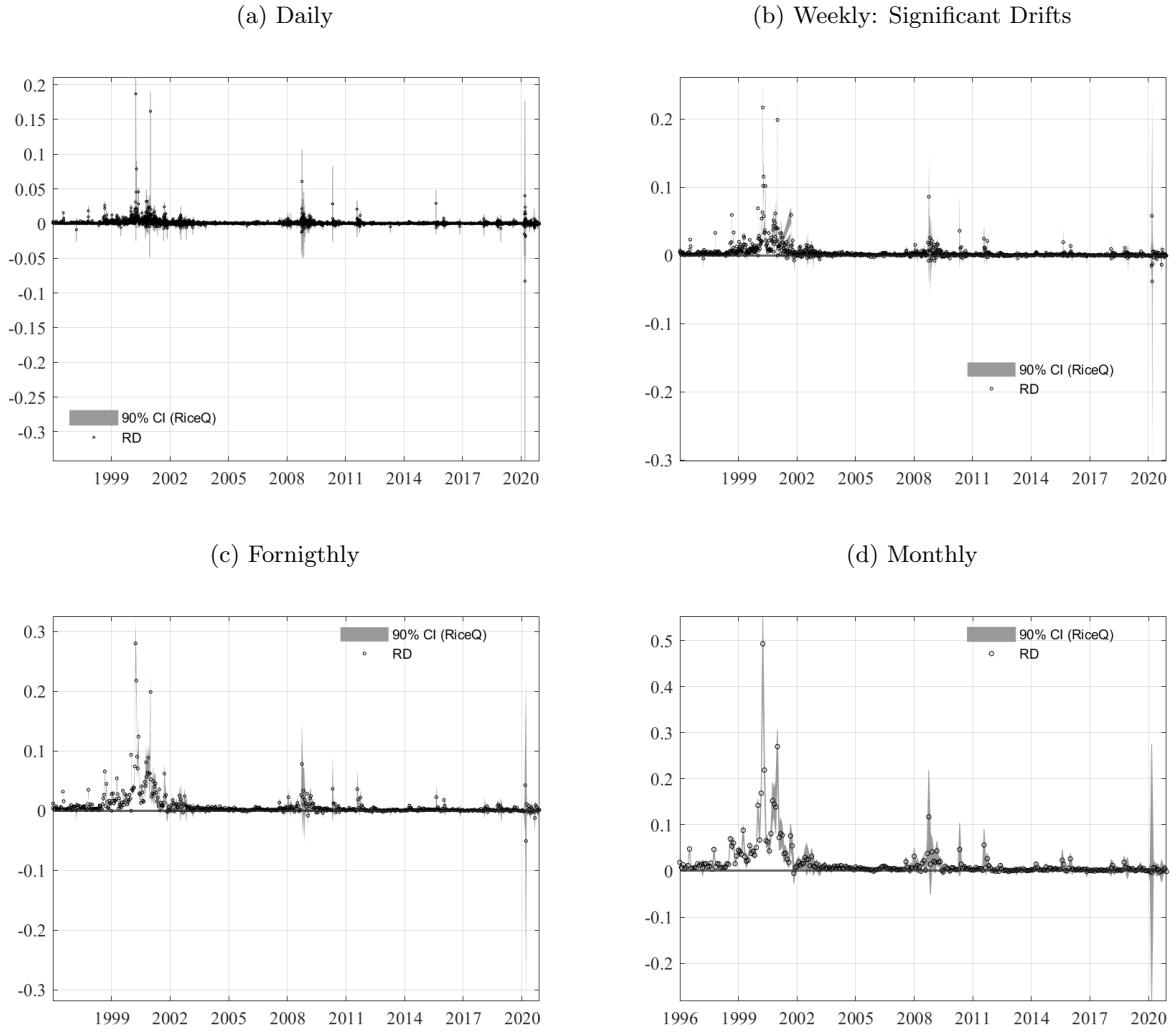
Figure 5: A tale of two confidence bands: Nasdaq composite index



model specifications were considered, including a standard Itō semimartingale process, an explosive linear drift diffusion model, a trending drift model, and a drift burst process. The realized drift measure was shown to provide a consistent estimator of integrated drift when the time span T extends to infinity at a sufficiently fast rate and the sampling interval Δ_n shrinks to zero. Under additional restrictions on T, RD follows a mixture normal distribution in the limit, with its conditional variance determined by the integrated quarticity.

The asymptotic variance of RD can be estimated using a drift-robust integrated quarticity estimator (RiceQ), which facilitates statistical inference. RiceQ is a consistent estimator of integrated quarticity under double asymptotics across all model specifications and exhibits smaller bias than realized quarticity in the presence of non-zero drift. Consistency for RiceQ requires that T does not diverge to infinity too rapidly in the case of explosive linear drift, trending drift, and drift burst processes. Building on this theory we proposed tests for detecting non-zero drifts using statistics constructed from RD and RiceQ. These tests are consistent and display good size and power performance in simulations. In finite samples

Figure 6: Realised drift estimates with confidence bands

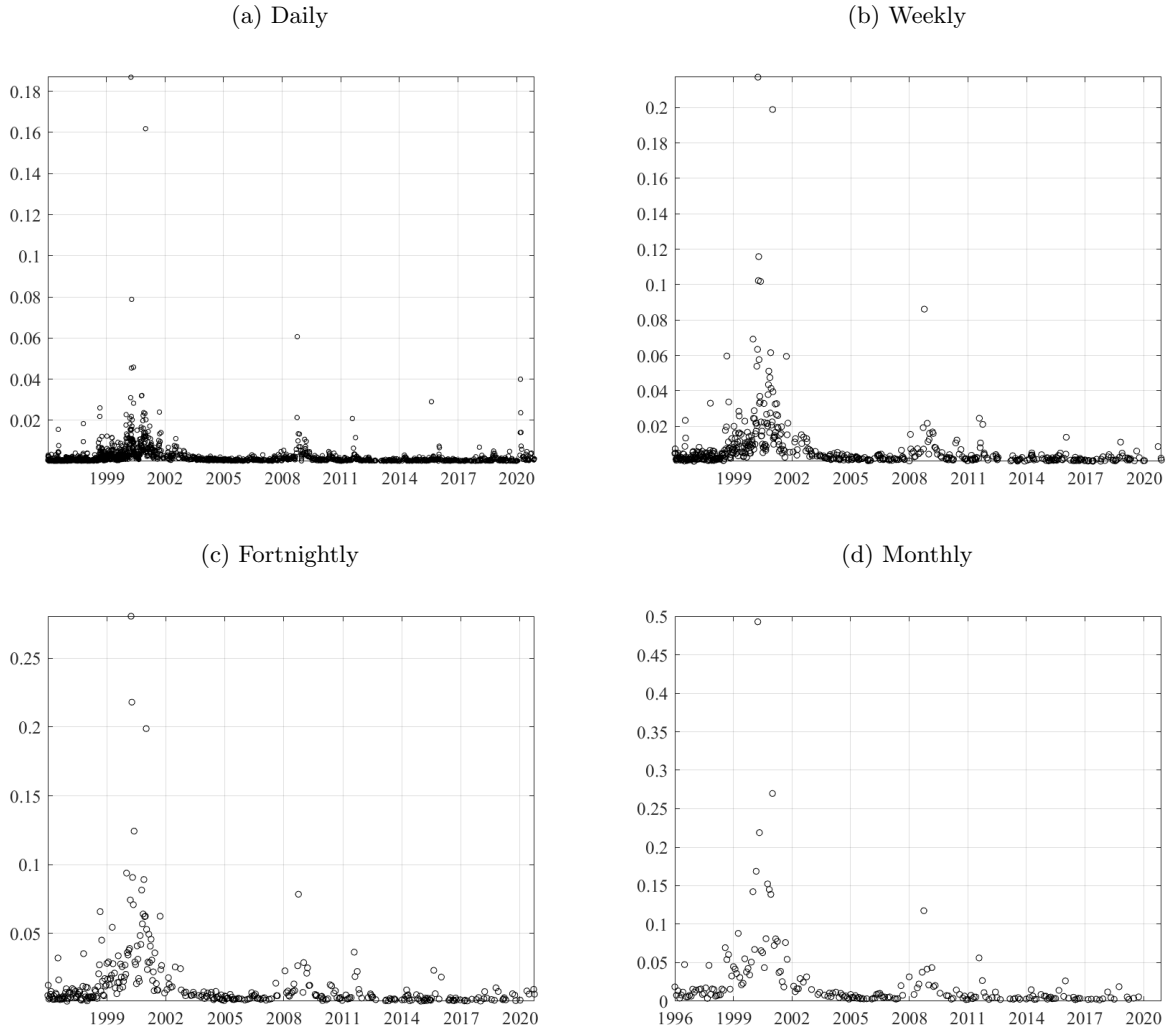


realized drift proves to be an effective measure of integrated drift, with both bias and standard deviation significantly decreasing as the time span of observation increases.

The drift test provides an alternative approach for identifying speculative bubbles and crises using intraday high frequency data. The empirical findings are consistent with our interpretation and the existing literature. The zero drift null hypothesis of the test is rejected frequently between 1996 and 2000, which is commonly agreed to contain bubble expansion and collapsing periods. The 2008 subprime mortgage crisis period and the 2020 stock market crash are identified as episodes of volatility bursting rather than drift explosion.

Realized drift offers a versatile tool with a wide range of potential applications in financial research and analysis. As demonstrated in Laurent et al. (2022), realized drift can be utilized for forecasting realized volatility. Understanding the drift component aids in accurately predicting the future volatility of financial assets. By incorporating the realized drift into volatility models, analysts and traders can make more precise volatility forecasts, which are essential for risk management and option pricing. Research, such as that conducted by Galí and Gambetti (2015), has shown that monetary policy can have a significant impact on stock market bubbles. Realized drift can play a role in measuring such bubbles and allows

Figure 7: Significant realised drift estimates



researchers to assess how monetary policy changes affect the drift component of asset prices, which is important for understanding the connection between central bank actions and financial market behavior. Realized drift can also assist in constructing optimal mean-variance portfolios by providing insight into the expected returns of different assets. By incorporating drift information, investors can make more informed decisions about asset allocation, balancing risk and return in achieving investment objectives.

References

Aalen, O. O. and H. K. Gjessing (2004). Survival models based on the Ornstein-Uhlenbeck process. *Lifetime data analysis 10*(4), 407–423. 17

Aït-Sahalia, Y., J. Jacod, and J. Li (2012). Testing for jumps in noisy high frequency data. *Journal of Econometrics 168*(2), 207–222. 16

Ait-Sahalia, Y., R. Kimmel, et al. (2007). Maximum likelihood estimation of stochastic volatility models. *Journal of financial economics 83*(2), 413–452. 16

Aït-Sahalia, Y., P. Mykland, and L. Zhang (2005). How often to sample a continuous-time process in the presence of market microstructure noise. *Review of Financial Studies 18*(2), 351–416. 2

Ait-Sahalia, Y. and J. Yu (2009). High frequency market microstructure noise estimates and liquidity measures. *Annals of Applied Statistics 3*, 422–457. 2

Andersen, T. and T. Bollerslev (1997). Intraday periodicity and volatility persistence in financial markets. *Journal of Empirical Finance 4*, 115–158. 2

Andersen, T. and T. Bollerslev (1998). Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. *International Economic Review 39*(4), 885–905. 1

Andersen, T., T. Bollerslev, and F. Diebold (2007). Roughing it up: Including jump components in the measurement, modeling, and forecasting of return volatility. *The Review of Economics and Statistics 89*, 701–720. 2

Andersen, T., Y. Li, V. Todorov, and B. Zhou (2021). Volatility measurement with pockets of extreme return persistence. *Journal of Econometrics*. 6, 47

Andersen, T. G., D. Dobrev, and E. Schaumburg (2012). Jump-robust volatility estimation using nearest neighbor truncation. *Journal of Econometrics 169*(1), 75–93. 47

Andersen, T. G., Y. Li, V. Todorov, and B. Zhou (2023). Volatility measurement with pockets of extreme return persistence. *Journal of Econometrics 237*(2), 105048. 47

Bandi, F. M. and P. C. B. Phillips (2003). Fully nonparametric estimation of scalar diffusion models. *Econometrica 71*(1), 241–283. 2

Bandi, F. M. and J. R. Russell (2006). Separating microstructure noise from volatility. *Journal of Financial Economics 79*(3), 655–692. 16

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2009). Realized kernels in practice: Trades and quotes. 1, 20

Barndorff-Nielsen, O. E. and N. Shephard (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 63*(2), 167–241. 17

Barndorff-Nielsen, O. E. and N. Shephard (2003). Realized power variation and stochastic volatility models. *Bernoulli 9*(2), 243–265. 12

Barndorff-Nielsen, O. E. and N. Shephard (2004). Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics 2*(1), 1–37. 1

Boudt, K., O. Kleen, and E. Sjørup (2021). Analyzing intraday financial data in r: The highfrequency package. *Available at SSRN 3917548*. 20

Cascaldi-Garcia, D. (2022). Forecast revisions as instruments for news shocks. *International Finance Discussion Paper* (1341). 2

Christensen, K., R. Oomen, and R. Renò (2020). The drift burst hypothesis. *Journal of Econometrics*. 16

Christensen, K., R. C. Oomen, and M. Podolskij (2014). Fact or friction: Jumps at ultra high frequency. *Journal of Financial Economics 114*(3), 576–599. 16

Christensen, K., R. C. A. Oomen, and R. Renò (2022). The drift burst hypothesis. *Journal of Econometrics 227*(2), 461–497. 2, 6

Christensen, K., M. Thyrsgaard, and B. Veliyev (2019). The realized empirical distribution function of stochastic variance with application to goodness-of-fit testing. *Journal of Econometrics 212*(2), 556–583. 16

Christoffersen, P., B. Feunou, K. Jacobs, and N. Meddahi (2014). The economic value of realized volatility: Using high-frequency returns for option valuation. *Journal of Financial and Quantitative Analysis 49*(3), 663–697. 2

Christoffersen, P. F. and F. X. Diebold (2000). How relevant is volatility forecasting for financial risk management? *Review of Economics and Statistics 82*(1), 12–22. 2

Fleming, J., C. Kirby, and B. Ostdiek (2003). The economic value of volatility timing using "realized" volatility. *Journal of Financial Economics 67*(3), 473–509. 2

Galí, J. and L. Gambetti (2015). The effects of monetary policy on stock market bubbles: Some evidence. *American Economic Journal: Macroeconomics 7*(1), 233–57. 23

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies 6*(2), 327–343. 16

Jacod, J. and P. Protter (2011a). *Discretization of processes*, Volume 67. Springer Science & Business Media. 28, 29, 31, 32, 35

Jacod, J. and P. Protter (2011b). *Discretization of processes*, Volume 67. Springer Science & Business Media. 46

Jagannathan, R. and T. Ma (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance 58*(4), 1651–1683. 2

Kolokolov, A., R. Renò, and P. Zoi (2023). Bumvu estimators. *Working paper*. 7

Laurent, S., R. Renò, and S. Shi (2022). Realized drift. *Available at SSRN 4084647*. 2, 6, 7, 12, 16, 21, 23, 32

Laurent, S. and S. Shi (2020). Volatility estimation and jump detection for drift–diffusion processes. *Journal of Econometrics 217*(2), 259–290. 2, 21

Laurent, S. and S. Shi (2022). Unit root test with high-frequency data. *Econometric Theory 38*(1), 113–171. 5

Lee, S. S. and P. A. Mykland (2008). Jumps in financial markets: A new nonparametric test and jump dynamics. *The Review of Financial Studies 21*(6), 2535–2563. 2

Li, Z. M. and O. Linton (2022). A ReMeDI for microstructure noise. *Econometrica 90*(1), 367–389. 7

Lo, A. W. and J. Wang (1995). Implementing option pricing models when asset returns are predictable. *The Journal of Finance 50*(1), 87–129. 17

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of financial economics 8*(4), 323–361. 2

Nicolato, E. and E. Venardos (2003). Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Mathematical Finance 13*(4), 445–466. 17

Phillips, P. C. B. (1987). Towards a unified asymptotic theory for autoregression. *Biometrika 74*(3), 535–547. 6

Phillips, P. C. B. and S. Shi (2019). Detecting financial collapse and ballooning sovereign risk. *Oxford Bulletin of Economics and Statistics 81*(6), 1336–1361. 4

Phillips, P. C. B., S. Shi, and J. Yu (2015a). Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500. *International Economic Review 56*(4), 1043–1078. 5

Phillips, P. C. B., S. Shi, and J. Yu (2015b). Testing for multiple bubbles: Limit theory of real-time detectors. *International Economic Review 56*(4), 1079–1134. 5

Phillips, P. C. B., Y. Wu, and J. Yu (2011). Explosive behavior in the 1990s NASDAQ: When did exuberance escalate asset values? *International Economic Review 52*(1), 201–226. 5, 21

Shi, S. and P. C. B. Phillips (2022). Econometric analysis of asset price bubbles. 5

Shi, S. and Y. Song (2016). Identifying speculative bubbles using an infinite hidden Markov model. *Journal of Financial Econometrics 14*(1), 159–184. 21

Tao, Y., P. C. B. Phillips, and J. Yu (2019). Random coefficient continuous systems: Testing for extreme sample path behavior. *Journal of Econometrics 209*(2), 208–237. 2

Taylor, S. and X. Xu (1997). The incremental volatility information in one million foreign exchange quotations. *Journal of Empirical Finance 4*, 317–340. 2

Wang, X. and J. Yu (2016). Double asymptotics for explosive continuous time models. *Journal of Econometrics 193*, 35–53. 17

Wang, X. and J. Yu (2023). Bubble testing under polynomial trends. *Econometrics Journal 26*(1), 25–44. 6

Zhang, L., P. A. Mykland, and Y. Aït-Sahalia (2005). A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association 100*(472), 1394–1411. 16, 18, 21

Zhou, Q. and J. Yu (2015). Asymptotic theory for linear diffusions under alternative sampling schemes. *Economics Letters 128*, 1–5. 17

A Realized Drift

This section collects derivations of the individual components to deliver the asymptotics of RD_T in the double asymptotic setting under the four model specificiations. The findings are then grouped in summary to provide proofs of the theorems.

A.1 Standard Model Specification

Proof. (1) Using the stochastic continuity and boundedness properties of μ_s,

$$D_{1,n} = \frac{1}{\Delta_n}\sum_{i=2}^{n}\left(\int_{t_{i-1}}^{t_i}\mu_s ds\right)\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right) = \Delta_n\sum_{i=2}^{n}\mu_{t_{i-1}}^2\left[1+o_p(1)\right] \sim_a \int_0^T \mu_s^2 ds = O_p(T). \tag{26}$$

The first term $D_{1,n}$ is asymptotically equivalent to the integrated drift $\int_0^T \mu_s^2 ds$, which is $O_p(T)$ under double asymptotics.

(2) For the second term

$$\begin{aligned} D_{2,n} &= -\frac{1}{\Delta_n}\sum_{i=1}^{n-1}\left(\int_{t_i}^{t_{i+1}}\mu_s ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right) - \frac{1}{\Delta_n}\sum_{i=2}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right) \\ &= -\frac{1}{\Delta_n}\sum_{i=2}^{n-1}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left(\int_{t_i}^{t_{i+1}}\mu_s ds + \int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right)\left[1+o_p(1)\right], \end{aligned}$$

where $o_p(1)$ captures the end effects and coefficient continuity. By the stochastic continuity and boundness properties of μ_s,

$$\begin{aligned} -\frac{1}{\Delta_n}\sum_{i=2}^{n-1}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left(\int_{t_i}^{t_{i+1}}\mu_s ds + \int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right) &= -2\sum_{i=2}^{n-1}\mu_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left[1+o_p(1)\right] \\ &= \sum_{i=2}^{n-1} v_{2,i}\left[1+o_p(1)\right], \end{aligned} \tag{27}$$

where $v_{2,i} = -2\mu_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)$. Now we apply Theorem 2.2.14 in Jacod and Protter (2011a). The crucial orders are $\sum_{i=2}^{n-1}\mathbb{E}_{t_{i-1}}\left[v_{2,i}\right] = 0$ and, by Ito isometry and the stochastic continuity and boundness properties of σ_s,

$$\sum_{i=2}^{n-1}\mathbb{E}_{t_{i-1}}\left[v_{2,i}^2\right] = 4\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\,\mathbb{E}_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^2 = 4\Delta_n\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2 \sim_a 4\int_0^T \mu_s^2\sigma_s^2 ds = O_p(T). \tag{28}$$

The second order is asymptotically equivalent to $4\int_0^T \mu_s^2\sigma_s^2 ds$. This implies that

$$T^{-1/2}D_{2,n} \to^d \mathcal{N}\left(0, 4\omega_{\mu\sigma}\right).$$

(3) For the third term, let $v_{3,i} = \Delta_n^{-1}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s dW_s\right)$ and write $D_{3,n} = \sum_{i=2}^{n} v_{3,i}$. The crucial order is $\sum_{i=2}^{n}\mathbb{E}_{i-2}\left[v_{3,i}\right] = 0$ and by the continuity of the coefficients,

$$\begin{aligned} \sum_{i=2}^{n}\mathbb{E}_{i-2}\left[v_{3,i}^2\right] &= \Delta_n^{-2}\sum_{i=2}^{n}\sigma_{t_{i-1}}^4\,\mathbb{E}_{i-2}\left[\left(\int_{t_{i-1}}^{t_i} dW_s\right)^2\left(\int_{t_{i-2}}^{t_{i-1}} dW_s\right)^2\right] \\ &= \sum_{i=2}^{n}\sigma_{t_{i-1}}^4 \sim_a \Delta_n^{-1}\int_0^T \sigma_s^4 ds = O_p(n). \end{aligned} \tag{29}$$

Thus, $D_{3,n}$ is $O_p\left(n^{1/2}\right)$ and

$$n^{-1/2}D_{3,n} \to^d \mathcal{MN}\left(0, \omega_{\sigma,4}\right). \tag{30}$$

■

A.2 Explosive Linear Drift

Proof. Under the explosive linear drift model, using result in (5), the drift component of returns is

$$\int_{t_{i-1}}^{t_i} \mu_s ds = \theta p_0 \int_{t_{i-1}}^{t_i} \exp(\theta s)\, ds\, [1 + o_p(1)] = p_0 \theta e^{\theta \tau_i} \Delta_n [1 + o_p(1)], \tag{31}$$

where $\tau_i \in [t_{i-1}, t_i]$. RD_T can be decomposed into three components

$$\mathsf{RD}_T = D_{1,n} + D_{2,n} + D_{3,n}, \tag{32}$$

where $D_{1,n}$, $D_{2,n}$, and $D_{3,n}$ are given in (11)-(13) with $\mu_s = \theta p_s$ and $\theta > 0$. (1) The first term

$$\begin{aligned} D_{1,n} &= \frac{1}{\Delta_n} \sum_{i=2}^{n} \left(\int_{t_{i-1}}^{t_i} \mu_s ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s ds \right) = p_0^2 \theta^2 \Delta_n \sum_{i=2}^{n} e^{\theta(\tau_i + \tau_{i-1})} [1 + o_p(1)] \\ &= p_0^2 \theta^2 \Delta_n \sum_{i=2}^{n} e^{2\theta t_i} [1 + o_p(1)] \sim_a p_0^2 \theta^2 \int_0^T e^{2\theta s} ds = \frac{1}{2} \theta p_0^2 \left(e^{2\theta T} - 1 \right) = O_p\left(e^{2\theta T} \right). \end{aligned}$$

The first term $D_{1,n}$ is asymptotically equivalent to the integrated drift $\int_0^T \mu_s^2 ds$, which is $O_p\left(e^{2\theta T}\right)$ under double asymptotics. (2) The second term can be rewritten as

$$D_{2,n} = -\frac{1}{\Delta_n} \sum_{i=2}^{n-1} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right) \left(\int_{t_i}^{t_{i+1}} \mu_s ds + \int_{t_{i-2}}^{t_{i-1}} \mu_s ds \right) [1 + o_p(1)],$$

where $o_p(1)$ captures the end effects and coefficient continuity. Using results in equation (31),

$$\begin{aligned} D_{2,n} &= p_0 \theta \sum_{i=2}^{n-1} \left(e^{\theta \tau_{i+1}} + e^{\theta \tau_{i-1}} \right) \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right) [1 + o_p(1)] \\ &= 2 p_0 \theta \sum_{i=2}^{n-1} e^{\theta t_{i+1}} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right) [1 + o_p(1)] = \sum_{i=2}^{n-1} v_{2,i} [1 + o_p(1)] \end{aligned} \tag{33}$$

where $v_{2,i} = 2 p_0 \theta e^{\theta t_{i+1}} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right)$. Now apply Theorem 2.2.14 in Jacod and Protter (2011a). The crucial orders are $\sum_{i=2}^{n-1} \mathbb{E}_{t_{i-1}}[v_{2,i}] = 0$ and, by Ito isometry and the stochastic continuity and boundness properties of σ_s,

$$\sum_{i=2}^{n-1} \mathbb{E}_{t_{i-1}}\left[v_{2,i}^2\right] = 4 \sum_{i=2}^{n-1} p_0^2 \theta^2 e^{2\theta t_{i+1}} \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 ds \right) \tag{34}$$

$$= 4 p_0^2 \theta^2 \sum_{i=2}^{n-1} e^{2\theta t_{i+1}} \sigma_{t_i}^2 \Delta_n [1 + o_p(1)] \tag{35}$$

$$\sim_a 4 p_0^2 \theta^2 \int_0^T e^{2\theta s} \sigma_s^2 ds \leq 4 p_0^2 \theta^2 e^{2\theta T} \int_0^T \sigma_s^2 ds = O_p\left(e^{2\theta T} T \right). \tag{36}$$

The second term $D_{2,n}$ is $O_p\left(e^{\theta T}\sqrt{T}\right)$ and

$$\frac{1}{e^{\theta T}\sqrt{T}} D_{2,n} \to^d \mathcal{MN}\left(0, 4 p_0^2 \theta^2 \int_0^T e^{2\theta(s-T)} \sigma_s^2 ds \right).$$

(3) From (30),

$$n^{-1/2}D_{3,n} \to^d \mathcal{MN}\left(0, \omega_{\sigma,4}\right).$$

The drift term $D_{1,n}$ therefore dominates and is followed by $D_{3,n}$ if

$$\frac{e^{\theta T}\sqrt{T}}{\sqrt{n}} = e^{\theta T}\Delta_n^{1/2} \to 0.$$

■

A.3 Trending Drift

Proof. Under this data generating process

$$\int_{t_{i-1}}^{t_i} \mu_s ds = \int_{t_{i-1}}^{t_i} \left(\theta_0 + \theta_1 s\right) ds = \frac{1}{2}\Delta_n \left(2\theta_0 + \theta_1 t_i + \theta_1 t_{i-1}\right) = \Delta_n \left(\theta_0 + \theta_1 t_i\right)\left[1 + o_p\left(1\right)\right].$$

The statistic RD_T can be decomposed as follows

$$\mathsf{RD}_T = D_{1,n} + D_{2,n} + D_{3,n}, \tag{37}$$

where $D_{1,n}$, $D_{2,n}$, and $D_{3,n}$ are given in (11)-(13) with $\mu_s = \theta_0 + \theta_1 s$. (1) The first term

$$\begin{aligned} D_{1,n} &= \frac{1}{\Delta_n}\sum_{i=2}^{n}\left(\int_{t_{i-1}}^{t_i}\mu_s ds\right)\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right) \\ &= \frac{1}{\Delta_n}\sum_{i=2}^{n}\Delta_n^2\left(\theta_0 + \theta_1 t_{i-1}\right)^2\left[1 + o_p\left(1\right)\right] \\ &= \Delta_n\theta_1^2\sum_{i=2}^{n}t_{i-1}^2\left[1 + o_p\left(1\right)\right] = \frac{1}{3}T^3\theta_1^2\left[1 + o_p\left(1\right)\right] = O\left(T^3\right), \end{aligned}$$

since $\sum_{i=2}^{n} t_{i-1}^2 = \Delta_n^2 \sum_{i=1}^{n-1} i^2 = O\left(n^3\Delta_n^2\right)$. The first term $D_{1,n}$ is $O_p\left(T^3\right)$and asymptotically equivalent to the integrated drift since

$$\int_0^T \mu_s^2 ds = \int_0^T \left(\theta_0 + \theta_1 s\right)^2 ds = \frac{1}{3}T^3\theta_1^2\left[1 + o\left(1\right)\right].$$

(2) The second term

$$D_{2,n} = \frac{1}{\Delta_n}\sum_{i=2}^{n-1}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left(\int_{t_i}^{t_{i+1}}\mu_s ds + \int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right)\left[1 + o_p\left(1\right)\right],$$

where $o_p\left(1\right)$ captures the end effects and coefficient continuity. Since

$$\int_{t_i}^{t_{i+1}}\mu_s ds + \int_{t_{i-2}}^{t_{i-1}}\mu_s ds = 2\Delta_n\left(\theta_0 + \theta_1 t_i\right)\left[1 + o_p\left(1\right)\right],$$

we have

$$D_{2,n} = \sum_{i=2}^{n-1} 2\Delta_n\left(\theta_0 + \theta_1 t_i\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)\left[1 + o_p\left(1\right)\right] = \sum_{i=2}^{n-1} v_{2,i}\left[1 + o_p\left(1\right)\right]$$

where $v_{2,i} = 2\Delta_n (\theta_0 + \theta_1 t_i) \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right)$. Now we apply Theorem 2.2.14 in Jacod and Protter (2011a). The crucial orders are $\sum_{i=2}^{n-1} \mathbb{E}_{t_{i-1}} [v_{2,i}] = 0$ and, by Ito isometry and the stochastic continuity and boundness properties of σ_s,

$$\begin{aligned}
\sum_{i=2}^{n-1} \mathbb{E}_{t_{i-1}} \left[v_{2,i}^2 \right] &= 4\Delta_n^2 \sum_{i=2}^{n-1} (\theta_0 + \theta_1 t_i)^2 \left(\int_{t_{i-1}}^{t_i} \sigma_s^2 ds \right) \\
&= 4\Delta_n^3 \sum_{i=2}^{n-1} (\theta_0 + \theta_1 t_i)^2 \sigma_{t_{i-1}}^2 [1 + o_p(1)] \\
&\sim_a 4\Delta_n^2 \int_0^T (\theta_0 + \theta_1 s)^2 \sigma_s^2 ds = O_p \left(T^3 \Delta_n^2 \right).
\end{aligned}$$

The second term $D_{2,n}$ is $O_p \left(\Delta_n T^{3/2} \right)$. (3) From (30),

$$n^{-1/2} D_{3,n} \to^d \mathcal{MN}(0, \omega_{\sigma,4}).$$

Therefore, under double asymptotics the drift term dominates when $T^{5/2} \Delta_n^{1/2} \to \infty$. Moreover, if $T \Delta_n^{3/2} \to 0$, $D_{3,n}$ dominates $D_{2,n}$ and

$$\frac{1}{\sqrt{n}} \left[\mathsf{RD}_T - \int_0^T \mu_s^2 ds \right] \xrightarrow{d} \mathcal{MN}(0, \omega_{\sigma,4}). \tag{38}$$

■

A.4 Drift Bursting

The following result is useful in establishing the asymptotics.

Lemma A.1 *Let $\xi_{n-i-k} \in [n-i-k, n-i-k+1]$ for $i = 1, 2, \ldots, n-k-1$ and $\xi_0 = (1-\alpha)^{1/\alpha}$ and $\xi'_{n-i-k} \in [n-i-k, n-i-k+1]$ with $i = 1, \ldots n-1$ and $\xi'_0 = (1-2\beta)^{1/(2\beta)}$.*

(1) $\int_{t_{i+k-1}}^{t_{i+k}} (1 - \frac{s}{T})^{-\alpha} ds = T^\alpha \Delta_n^{1-\alpha} \xi_{n-i-k}^{-\alpha}$;

(2) $\int_{t_{i+k-1}}^{t_{i+k}} (1 - \frac{s}{T})^{-2\beta} ds = T^{2\beta} \Delta_n^{1-2\beta} \xi'^{-2\beta}_{n-i-k}$.

Proof. By the mean-value theorem, we have

$$\begin{aligned}
&\int_{t_{i+k-1}}^{t_{i+k}} (1 - \frac{s}{T})^{-\alpha} ds = T^\alpha \int_{t_{i+k-1}}^{t_{i+k}} (T-s)^{-\alpha} ds = -T^\alpha \frac{1}{1-\alpha} \left[(T - t_{i+k})^{1-\alpha} - (T - t_{i+k-1})^{1-\alpha} \right] \\
&= -T^\alpha \Delta_n^{1-\alpha} \frac{1}{1-\alpha} \left[(n-i-k)^{1-\alpha} - (n-i-k+1)^{1-\alpha} \right] = T^\alpha \Delta_n^{1-\alpha} \xi_{n-i-k}^{-\alpha},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{i+k-1}}^{t_{i+k}} (1 - \frac{s}{T})^{-2\beta} ds = T^{2\beta} \int_{t_{i+k-1}}^{t_{i+k}} (T-s)^{-2\beta} ds = -T^{2\beta} \frac{1}{1-2\beta} \left[(T - t_{i+k})^{1-2\beta} - (T - t_{i+k-1})^{1-2\beta} \right] \\
&= -T^{2\beta} \frac{1}{1-2\beta} \Delta_n^{1-2\beta} \left[(n-i-k)^{1-2\beta} - (n-i-k+1)^{1-2\beta} \right] = T^{2\beta} \Delta_n^{1-2\beta} \xi'^{-2\beta}_{n-i-k}.
\end{aligned}$$

■

Proof of Theorem 3.4.

The derivations below follow closely of those in Laurent et al. (2022). We assume here that $\alpha \in [0, 1/2)$, $\beta \in [0, 1/4)$, $T \to \infty$, and $\Delta_n \to 0$. (1) Using the stochastic continuity and bounded properties of μ_s combined with the results of Lemma A.1,

$$\begin{aligned}
\tilde{D}_{1,n} &= \frac{1}{\Delta_n}\sum_{i=2}^{n}\left(\int_{t_{i-1}}^{t_i}\mu_s(1-\frac{s}{T})^{-\alpha}ds\right)\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s(1-\frac{s}{T})^{-\alpha}ds\right)\\
&= \frac{1}{\Delta_n}\sum_{i=2}^{n}\mu_{t_{i-1}}^2\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\alpha}ds\right)\left(\int_{t_{i-2}}^{t_{i-1}}(1-\frac{s}{T})^{-\alpha}ds\right)\\
&= T^{2\alpha}\Delta_n^{1-2\alpha}\sum_{i=2}^{n}\mu_{t_{i-1}}^2\xi_{n-i}^{-\alpha}\xi_{n-i+1}^{-\alpha}.
\end{aligned}$$

Then, by simple Riemann sum arguments

$$D_{1,n} = \Delta_n\sum_{j=1}^{n-1}\mu_{t_{n-j}}^2\left(\frac{\xi_{n-i}\Delta_n}{T}\right)^{-\alpha}\left(\frac{\xi_{n-i+1}\Delta_n}{T}\right)^{-\alpha} \sim_a \int_0^T \mu_s^2(1-\frac{s}{T})^{-2\alpha}ds = O_p(T)$$

since $\int_0^1(1-u)^{-2\alpha}du$ is convergent when $\alpha < 1/2$.

(2) By reorganising terms and using the stochastic continuity and boundedness properties of μ_s and σ_s with the results of Lemma A.1, the second term is

$$\begin{aligned}
\tilde{D}_{2,n} &= -\frac{1}{\Delta_n}\sum_{i=1}^{n-1}\left(\int_{t_i}^{t_{i+1}}\mu_s(1-\frac{s}{T})^{-\alpha}ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s(1-\frac{s}{T})^{-\beta}dW_s\right)\\
&\quad -\frac{1}{\Delta_n}\sum_{i=2}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s(1-\frac{s}{T})^{-\beta}dW_s\right)\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s(1-\frac{s}{T})^{-\alpha}ds\right)\\
&= -\frac{1}{\Delta_n}\sum_{i=2}^{n-1}\mu_{t_{i-1}}\sigma_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\beta}dW_s\right)\left(\int_{t_i}^{t_{i+1}}(1-\frac{s}{T})^{-\alpha}ds+\int_{t_{i-2}}^{t_{i-1}}(1-\frac{s}{T})^{-\alpha}ds\right)[1+o_p(1)]\\
&= -T^{\alpha}\Delta_n^{-\alpha}\sum_{i=2}^{n-1}\mu_{t_{i-1}}\sigma_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\beta}dW_s\right)\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)[1+o_p(1)],
\end{aligned}$$

where the $o_p(1)$ term takes care of the end effect and the continuity of the coefficients. Write $\tilde{D}_{2,n} = \sum_{i=2}^{n-1}\tilde{v}_{2,t}[1+o_p(1)]$ with

$$\tilde{v}_{2,i} = -T^{\alpha}\Delta_n^{-\alpha}\mu_{t_{i-1}}\sigma_{t_{i-1}}\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\beta}dW_s\right).$$

Now apply Jacod and Protter (2011a, Theorem 2.2.14). The conditional mean is $\sum_{i=2}^{n-1}E_{i-1}[\tilde{v}_{2,i}] = 0$ and the conditional variance is

$$\begin{aligned}
\sum_{i=2}^{n-1}E_{i-1}\left[\tilde{v}_{2,i}^2\right] &= \sum_{i=2}^{n-1}T^{2\alpha}\Delta_n^{-2\alpha}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2E_{i-1}\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\beta}dW_s\right)^2\\
&= \sum_{i=2}^{n-1}T^{2\alpha}\Delta_n^{-2\alpha}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-2\beta}ds
\end{aligned}$$

$$= \quad T^{2(\alpha+\beta)}\Delta_n^{1-2\alpha-2\beta}\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}, \tag{39}$$

using Ito isometry and results in Lemma A.1(2). The order of (39) depends on the value of $\alpha+\beta$ and the cases are separated as follows:

- When $\alpha+\beta<1/2$,

$$\begin{aligned}\sum_{i=2}^{n-1}E_{i-1}\left[\tilde{v}_{2,i}^2\right] &= \Delta_n\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left[\left(\frac{\xi_{n-i-1}\Delta_n}{T}\right)^{-\alpha}+\left(\frac{\xi_{n-i+1}\Delta_n}{T}\right)^{-\alpha}\right]^2\left(\frac{\xi_{n-i}'\Delta_n}{T}\right)^{-2\beta}\\ &\sim_a 4\int_0^T\mu_s^2\sigma_s^2(1-\frac{s}{T})^{-2(\alpha+\beta)}ds=O_p\left(T\right).\end{aligned}$$

- When $\alpha+\beta>1/2$, $\zeta_{\alpha,\beta}^{\mu^2\sigma^2}\equiv\lim_{n\to\infty}\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}<\infty$ since μ_s and σ_s are bounded and

$$\begin{aligned}S_n &= \sum_{i=2}^{n-1}\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}=\left(\xi_0^{-\alpha}+\xi_2^{-\alpha}\right)^2\xi_1'^{-2\beta}+\sum_{i=2}^{n-2}\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}\\ &\le C+4\sum_{i=2}^{n-2}(n-i-1)^{-2(\alpha+\beta)}=C+4\sum_{j=1}^{n-3}j^{-2(\alpha+\beta)},\end{aligned}$$

  which is bounded given that $\sum_{j=1}^{\infty}j^{-2(\alpha+\beta)}<\infty$ is convergent. Therefore,

$$\begin{aligned}\sum_{i=2}^{n-1}E_{i-1}\left[\tilde{v}_{2,i}^2\right] &= T^{2(\alpha+\beta)}\Delta_n^{1-2\alpha-2\beta}\sum_{i=2}^{n-1}\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^2\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}\\ &\sim_a T^{2(\alpha+\beta)}\Delta_n^{1-2\alpha-2\beta}\zeta_{\alpha,\beta}^{\mu^2\sigma^2}=O_p\left(T^{2(\alpha+\beta)}\Delta_n^{1-2\alpha-2\beta}\right).\end{aligned} \tag{40}$$

- When $\alpha+\beta=1/2$, neither S_n nor $\int_0^1(1-u)^{-1}du$ is convergent. Using the fact that $j\le\xi_j\le j+1$, and noticing that $\xi_0=(1-\alpha)^{1/\alpha}$ and $\xi_0'=(1-2\beta)^{1/(2\beta)}$, we have

$$4\sum_{j=2}^{n-1}\frac{1}{j}\le\sum_{i=2}^{n-1}\left(\xi_{n-i-1}^{-\alpha}+\xi_{n-i+1}^{-\alpha}\right)^2\xi_{n-i}'^{-2\beta}\le C+4\sum_{j=1}^{n-3}\frac{1}{j},$$

  which is $O(\log(n))$. Therefore, $\sum_{i=2}^{n-1}E_{i-1}\left[\tilde{v}_{2,i}^2\right]=O_p\left(T\log n\right)$.

Consequently, we have

$$\tilde{D}_{2,n}=\begin{cases}O_p\left(\sqrt{T}\right) & \text{if } \alpha+\beta<1/2\\ O_p\left(T^{1/2}\sqrt{\log n}\right) & \text{if } \alpha+\beta=1/2\\ O_p\left(T^{\alpha+\beta}\Delta_n^{1/2-\alpha-\beta}\right) & \text{if } \alpha+\beta>1/2\end{cases}.$$

(3) Let $\tilde{v}_{3,i}=\Delta_n^{-1}\left(\int_{t_{i-1}}^{t_i}\sigma_s(1-\frac{s}{T})^{-\beta}dW_s\right)\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s(1-\frac{s}{T})^{-\beta}dW_s\right)$ and write $\tilde{D}_{3,n}=\sum_{i=2}^{n}v_{3,i}$. The conditional mean is $\sum_{i=2}^{n}\mathbb{E}_{i-2}\left[\tilde{v}_{3,i}\right]=0$ and, by the continuity of the coefficients, Ito isometry, and Lemma A.1, the conditional variance is

$$\sum_{i=2}^{n}\mathbb{E}_{i-2}\left[\tilde{v}_{3,i}^2\right]=\Delta_n^{-2}\sum_{i=2}^{n}\sigma_{t_{i-1}}^4\,\mathbb{E}_{i-2}\left[\left(\int_{t_{i-1}}^{t_i}(1-\frac{s}{T})^{-\beta}dW_s\right)^2\left(\int_{t_{i-2}}^{t_{i-1}}(1-\frac{s}{T})^{-\beta}dW_s\right)^2\right]$$

$$= \Delta_n^{-2} \sum_{i=2}^{n} \sigma_{t_{i-1}}^4 \left(\int_{t_{i-1}}^{t_i} (1-\frac{s}{T})^{-2\beta} ds \right) \left(\int_{t_{i-2}}^{t_{i-1}} (1-\frac{s}{T})^{-2\beta} ds \right) = T^{4\beta} \Delta_n^{-4\beta} \sum_{i=2}^{n} \sigma_{t_{i-1}}^4 \xi_{n-i}'^{-2\beta} \xi_{n-i+1}'^{-2\beta}$$

$$= \sum_{i=2}^{n} \sigma_{t_{i-1}}^4 \left(\frac{\xi_{n-i}' \Delta_n}{T} \right)^{-2\beta} \left(\frac{\xi_{n-i+1}' \Delta_n}{T} \right)^{-2\beta} \sim_a \Delta_n^{-1} \int_0^T \sigma_s^4 \left(1-\frac{s}{T}\right)^{-4\beta} ds = O_p(n),$$

since $\int_0^1 (1-u)^{-4\beta} du$ is convergent when $\beta < 1/4$. These results imply that $\tilde{D}_{3,n}$ is $O_p\left(n^{1/2}\right)$ and

$$n^{-1/2} \tilde{D}_{3,n} \sim_a \mathcal{N}\left(0, \omega_{\sigma^4,\beta}\right),$$

where $\omega_{\sigma^4,\beta} \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma_s^4 \left(1-\frac{s}{T}\right)^{-4\beta} ds$.

Therefore, the asymptotic properties of RD_T are as follows.

- When $\alpha+\beta < 1/2$: if $T\Delta_n \to \infty$, $\tilde{D}_{1,n}$ dominates $\tilde{D}_{2,n}$ and $\tilde{D}_{3,n}$ and

$$n^{-1/2} \left[\mathsf{RD}_T - \int_0^T \mu_s^2 (1-\frac{s}{T})^{-2\alpha} ds \right] \to_d \mathcal{N}\left(0, \omega_{\sigma^4,\beta}\right).$$

- When $\alpha+\beta = 1/2$: if $T\Delta_n \to \infty$, we have

$$\frac{\tilde{D}_{1,n}}{\tilde{D}_{3,n}} \quad : \quad \frac{T}{T^{1/2}\Delta_n^{-1/2}} = (T\Delta_n)^{1/2} \to \infty.$$
$$\frac{\tilde{D}_{3,n}}{\tilde{D}_{2,n}} \quad : \quad \frac{T^{1/2}\Delta_n^{-1/2}}{T^{1/2}\sqrt{\log n}} = \frac{\Delta_n^{-1/2}}{\sqrt{\log n}} \to \infty.$$

  and hence

$$n^{-1/2} \left[\mathsf{RD}_T - \int_0^T \mu_s^2 (1-\frac{s}{T})^{-2\alpha} ds \right] \to_d \mathcal{N}\left(0, \omega_{\sigma^4,\beta}\right).$$

- When $\alpha+\beta > 1/2$: if $T\Delta_n \to \infty$, RD_T is dominated by the drift term $\tilde{D}_{1,n}$ since

$$\frac{\tilde{D}_{1,n}}{\tilde{D}_{3,n}} \quad : \quad \frac{T}{T^{1/2}\Delta_n^{-1/2}} = (T\Delta_n)^{1/2} \to \infty,$$
$$\frac{\tilde{D}_{1,n}}{\tilde{D}_{2,n}} \quad : \quad \frac{T}{T^{\alpha+\beta}\Delta_n^{1/2-\alpha-\beta}} = T^{1-(\alpha+\beta)} \Delta_n^{\alpha+\beta-1/2} \to \infty.$$

  Moreover, if, in addition,

$$\frac{\tilde{D}_{3,n}}{\tilde{D}_{2,n}} : \frac{T^{1/2}\Delta_n^{-1/2}}{T^{\alpha+\beta}\Delta_n^{1/2-\alpha-\beta}} = T^{1/2-(\alpha+\beta)} \Delta_n^{-1+\alpha+\beta} \to \infty, \tag{41}$$

  we have the following limiting distribution of RD_T:

$$n^{-1/2} \left[\mathsf{RD}_T - \int_0^T \mu_s^2 (1-\frac{s}{T})^{-2\alpha} ds \right] \to_d \mathcal{N}\left(0, \omega_{\sigma^4,\beta}\right).$$

  Let $T = cT^{-\psi}$. These two conditions (i.e., $T\Delta_n \to \infty$ and $T^{1/2-(\alpha+\beta)}\Delta_n^{-1+\alpha+\beta} \to \infty$) imply that

$$1 < \psi < \frac{1-(\alpha+\beta)}{\alpha+\beta-1/2}. \tag{42}$$

■

B Quarticity Estimators

This section collects proofs for the consistency of RiceQ_T in the double asymptotic setting under the four model specifications, along with double asymptotics for RQ_T under the standard Itō semimartingale specification.

B.1 Standard Model Specification

We prove consistency of RQ_T and RiceQ_T under the standard model specification (1). Section B.1.1 is for RQ_T and Section B.1.2 is for RiceQ_T. Let $U_{n,i} = \int_{t_{i-1}}^{t_i} \mu_s ds$ and $V_{n,i} = \int_{t_{i-1}}^{t_i} \sigma_s dW_s$.

B.1.1 RQ_T

Proof. Under the model specification (1),

$$r_{n,i}^4 = (U_{n,i} + V_{n,i})^4 = U_{n,i}^4 + V_{n,i}^4 + 4U_{n,i}^3 V_{n,i} + 6U_{n,i}^2 V_{n,i}^2 + 4U_{n,i} V_{n,i}^3$$

using the binomial theorem. The realized quarticity can be rewritten as

$$\mathsf{RQ} = \frac{1}{3\Delta_n} \sum_{i=1}^n r_{n,i}^4 = Q_{1,n} + Q_{2,n} + Q_{3,n} + Q_{4,n} + Q_{5,n},$$

where

$$\begin{array}{lll} Q_{1,n} \equiv \frac{1}{3}\Delta_n^{-1} \sum_{i=1}^n U_{n,i}^4, & Q_{2,n} \equiv \frac{4}{3}\Delta_n^{-1} \sum_{i=1}^n U_{n,i}^3 V_{n,i}, & Q_{3,n} \equiv 2\Delta_n^{-1} \sum_{i=1}^n U_{n,i}^2 V_{n,i}^2, \\ Q_{4,n} \equiv \frac{4}{3}\Delta_n^{-1} \sum_{i=1}^n U_{n,i} V_{n,i}^3, & Q_{5,n} \equiv \frac{1}{3}\Delta_n^{-1} \sum_{i=1}^n V_{n,i}^4. & \end{array}$$

(1) Using the stochastic continuity and bounded properties of μ_s,

$$Q_{1,n} = \frac{1}{3\Delta_n} \sum_{i=1}^n U_{n,i}^4 = \frac{1}{3}\Delta_n^3 \sum_{i=1}^n \mu_{t_{i-1}}^4 \left[1 + o_p(1)\right] = O_p\left(T\Delta_n^2\right).$$

(2) For $Q_{2,n}$, let $\eta_{2,i} = \frac{4}{3}\Delta_n^{-1} \left(\int_{t_{i-1}}^{t_i} \mu_s ds\right)^3 \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)$ so that $Q_{2,n} = \sum_{i=1}^n \eta_{2,i}$. Now apply Theorem 2.2.14 in Jacod and Protter (2011a) again. The crucial orders are: first $\sum_{i=1}^n \mathbb{E}_{t_{i-1}}\left[\eta_{2,i}\right] = 0$ and second, by Ito isometry and the stochastic continuity and boundness properties of σ_s, the conditional variance is

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}_{t_{i-1}} \eta_{2,i}^2 &= \frac{16}{9}\Delta_n^4 \sum_{i=1}^n \mu_{t_{i-1}}^6 \mathbb{E}_{t_{i-1}} \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 = \frac{16}{9}\Delta_n^5 \sum_{i=1}^n \mu_{t_{i-1}}^6 \sigma_{t_{i-1}}^2 \\ &\sim_a \frac{16}{9}\Delta_n^4 \int_0^T \mu_s^6 \sigma_s^2 ds = O_p\left(T\Delta_n^4\right). \end{aligned}$$

Therefore, $Q_{2,n}$ is $O_p\left(T^{1/2}\Delta_n^2\right)$ under the double asymptotic setting.
(3) For $Q_{3,n}$,

$$\begin{aligned} 2\Delta_n^{-1} \sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} \mu_s ds\right)^2 \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 &= 2\Delta_n \sum_{i=1}^n \mu_{t_{i-1}}^2 \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s\right)^2 \left[1 + o_p(1)\right] \\ &= 2\Delta_n^2 \sum_{i=1}^n \mu_{t_{i-1}}^2 \sigma_{t_{i-1}}^2 \left[1 + o_p(1)\right] \sim_a 2\Delta_n \int_0^T \mu_s^2 \sigma_s^2 ds \end{aligned}$$

since, from the properties of the stochastic integral,

$$\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^2=\int_{t_{i-1}}^{t_i}\sigma_s^2 ds+2\int_{t_{i-1}}^{t_i}\sigma_s\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)dW_s\sim_a\int_{t_{i-1}}^{t_i}\sigma_s^2 ds\sim_a\Delta_n\sigma_{t_{i-1}}^2.$$

Therefore, it is $O_p\left(T\Delta_n\right)$.

(4) For $Q_{4,n}$, let $\eta_{4,i}=\frac{4}{3}\Delta_n^{-1}\left(\int_{t_{i-1}}^{t_i}\mu_s ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^3$ so that $Q_{4,n}=\sum_{i=1}^n\eta_{4,i}$. The first order $\sum_{i=1}^n\mathbb{E}_{t_{i-1}}\eta_{4,i}=0$. By Ito's lemma,

$$\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^6=15\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^4\sigma_s^2 ds+6\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^5\sigma_s dW_s,$$

$$\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^4=4\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^3\sigma_s dW_s+6\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^2\sigma_s^2 ds,$$

and hence

$$\begin{aligned}\mathbb{E}_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^6 &= \mathbb{E}_{t_{i-1}}\left(15\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^4\sigma_s^2 ds+6\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^5\sigma_s dW_s\right)\\ &= 15\int_{t_{i-1}}^{t_i}\mathbb{E}_{t_{i-1}}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^4\sigma_t^2 dt=15\int_{t_{i-1}}^{t_i}\left[6\int_{t_{i-1}}^{s}\mathbb{E}_{t_{i-1}}\left(\int_{t_{i-1}}^{v}\sigma_u dW_u\right)^2\sigma_v^2 dv\right]\sigma_s^2 ds\\ &= 90\int_{t_{i-1}}^{t_i}\left[\int_{t_{i-1}}^{s}\left(\int_{t_{i-1}}^{v}\sigma_u^2 du\right)\sigma_v^2 dv\right]\sigma_s^2 ds=15\Delta_n^3\sigma_{t_{i-1}}^6\end{aligned}$$

It follows that the conditional variance is

$$\begin{aligned}\sum_{i=1}^n\mathbb{E}_{t_{i-1}}\eta_{4,i}^2 &= \frac{16}{9}\Delta_n^{-2}\sum_{i=1}^n\left(\int_{t_{i-1}}^{t_i}\mu_s ds\right)^2\mathbb{E}_{t_{i-1}}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^6\\ &= \frac{80}{3}\Delta_n^3\sum_{i=1}^n\mu_{t_{i-1}}^2\sigma_{t_{i-1}}^6\left[1+o_p\left(1\right)\right]=O_p\left(T\Delta_n^2\right).\end{aligned}$$

Therefore, $Q_{4,n}=O_p\left(T^{1/2}\Delta_n\right)$.

(5) The last term is, by Ito's lemma, integration by parts and the properties of σ_s,

$$\begin{aligned}Q_{5,n} &\equiv \frac{1}{3}\Delta_n^{-1}\sum_{i=1}^n\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^4\\ &= \frac{1}{3}\Delta_n^{-1}\sum_{i=1}^n\left[4\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^3\sigma_s dW_s+6\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^2\sigma_s^2 ds\right]\left[1+o_p\left(1\right)\right]\\ &= \frac{1}{3}\Delta_n^{-1}\sum_{i=1}^n\left[6\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u^2 du\right)^2\sigma_s^2 ds\right]\left[1+o_p\left(1\right)\right]\\ &= \Delta_n^{-1}\sum_{i=1}^n\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 ds\right)^2\left[1+o_p\left(1\right)\right]=\sum_{i=1}^n\sigma_{t_{i-1}}^4\Delta_n\left[1+o_p\left(1\right)\right]\sim_a\int_0^T\sigma_s^4 ds,\end{aligned}$$

which is $O_p(T)$ under the double asymptotic setting. Evidently, RQ_T is dominated by $Q_{5,n}$, followed by $Q_{3,n}$. Hence

$$\mathsf{RQ}_T = Q_{5,n}\left[1 + o_p(1)\right] \sim_a \int_0^T \sigma_s^4 ds,$$

and

$$\mathsf{RQ}_T - Q_{5,n} = Q_{3,n}\left[1 + o_p(1)\right] = O_p(T\Delta_n).$$

■

B.1.2 RiceQ_T

Proof. We derive the consistency of RiceQ_T. Under the model specification,

$$\begin{aligned}
\mathsf{RiceQ}_T &= \sum_{i=3}^{n} \frac{1}{6\Delta_n}\left(r_{n,i} - r_{n,i-1}\right)^2\left(r_{n,i-1} - r_{n,i-2}\right)^2 \\
&= \sum_{i=3}^{n} \frac{1}{6\Delta_n}\left(U_{n,i} - U_{n,i-1} + V_{n,i} - V_{n,i-1}\right)^2\left(U_{n,i-1} - U_{n,i-2} + V_{n,i-1} - V_{n,i-2}\right)^2 \\
&= \left(\tilde{Q}_{1,n} + \tilde{Q}_{2,n} + \tilde{Q}_{3,n}\right)\left[1 + o_p(1)\right], \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_{1,n} = {} & \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^4 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 U_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-2}^2 U_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 U_{n,i-2}^2 + \frac{2}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 U_{n,i-2} U_{n,i} \\
& - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^3 U_{n,i-2} - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1} U_{n,i-2}^2 U_{n,i} - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1} U_{n,i-2} U_{n,i}^2 - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^3 U_{n,i}, \qquad (44)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{2,n} = {} & \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 V_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^2 U_{n,i}^2 + \frac{1}{6\Delta_{.n}}\sum_{i=3}^{n} U_{n,i-2}^2 V_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-2}^2 U_{n,i}^2 \\
& + \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 V_{n,i-2}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-2}^2 V_{n,i-1}^2 + \frac{1}{\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2 V_{n,i-1}^2 + \frac{2}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-2} V_{n,i-1}^2 U_{n,i} \\
& - \frac{1}{\Delta_n}\sum_{i=3}^{n} U_{n,i-1} V_{n,i-1}^2 U_{n,i} - \frac{1}{\Delta_n}\sum_{i=3}^{n} U_{n,i-1} U_{n,i-2} V_{n,i-1}^2 - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1} V_{n,i-2}^2 U_{n,i} - \frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1} U_{n,i-2} V_{n,i}^2, \qquad (45)
\end{aligned}$$

and

$$\tilde{Q}_{3,n} = \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^2 V_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-2}^2 V_{n,i}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^2 V_{n,i-2}^2 + \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^4. \qquad (46)$$

(1) We first investigate the asymptotic order of $Q_{1,n}$. Rewrite

$$\begin{aligned}\tilde{Q}_{1,n} &= \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^3\left(U_{n,i-1}-U_{n,i-2}\right)+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}U_{n,i}^2\left(U_{n,i-1}-U_{n,i-2}\right)\\ &+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-2}U_{n,i}^2\left(U_{n,i-2}-U_{n,i-1}\right)+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}U_{n,i-2}^2\left(U_{n,i-1}-U_{n,i}\right)\\ &+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2U_{n,i-2}\left(U_{n,i}-U_{n,i-1}\right)+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}U_{n,i}U_{n,i-2}\left(U_{n,i-1}-U_{n,i-2}\right)\\ &+\frac{1}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^2U_{n,i}\left(U_{n,i-2}-U_{n,i-1}\right).\end{aligned} \tag{47}$$

By the Hölder continuity of μ_s,

$$\left|\int_{t_{i-2}}^{t_{i-1}}\mu_s ds-\int_{t_{i-3}}^{t_{i-2}}\mu_s ds\right|\leq\int_{t_{i-2}}^{t_{i-1}}\left|\mu_s-\mu_{s-\Delta_n}\right|ds\leq K\Delta_n^{1+\Gamma/2}=O_p\left(\Delta_n^{1+\Gamma/2}\right). \tag{48}$$

The first term in (47) has the following properties:

$$\begin{aligned}&\left|\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^3\left(U_{n,i-1}-U_{n,i-2}\right)\right|=\left|\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right)^3\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds-\int_{t_{i-3}}^{t_{i-2}}\mu_s ds\right)\right|\\ \leq\ &\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\left|\mu_s\right|ds\right)^3\int_{t_{i-2}}^{t_{i-1}}\left|\mu_s-\mu_{s-k\Delta_n}\right|ds\\ \leq\ &K\Delta_n^{3+\Gamma/2}\sum_{i=3}^{n}\left|\mu_{t_{i-1}}\right|^3\sim_a\Delta_n^{2+\Gamma/2}\int_0^T\left|\mu_s\right|^3ds=O_p\left(T\Delta_n^{2+\Gamma/2}\right),\end{aligned}$$

using the result in (48) and the boundedness of μ_s. Similarly, we can show that all remaining terms in (47) are of order $O_p\left(T\Delta_n^{2+\Gamma/2}\right)$. Therefore,

$$\tilde{Q}_{1,n}=O_p\left(T\Delta_n^{2+\Gamma/2}\right).$$

(2) Next, we investigate the asymptotic properties of $\tilde{Q}_{2,n}$. Rewrite

$$\begin{aligned}\tilde{Q}_{2,n} &= \frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1}-U_{n,i-2}\right)V_{n,i}^2+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i}\left(U_{n,i}-U_{n,i-1}\right)V_{n,i-1}^2\\ &+\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(U_{n,i-2}-U_{n,i-1}\right)U_{n,i-2}V_{n,i}^2+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i}\left(U_{n,i}-U_{n,i-1}\right)V_{n,i-2}^2\\ &+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1}-U_{n,i}\right)V_{n,i-2}^2+\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(U_{n,i-2}-U_{n,i-1}\right)U_{n,i-2}V_{n,i-1}^2\\ &+\frac{5}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1}-U_{n,i}\right)V_{n,i-1}^2+\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1}-U_{n,i-2}\right)V_{n,i-1}^2\\ &+\frac{2}{3\Delta_n}\sum_{i=3}^{n} U_{n,i-2}\left(U_{n,i}-U_{n,i-1}\right)V_{n,i-1}^2.\end{aligned} \tag{49}$$

The first term in (49) has the following properties:

$$\begin{aligned}
& \frac{1}{6\Delta_n}\left|\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1}-U_{n,i-2}\right)V_{n,i}^2\right| \leq \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left|U_{n,i-1}\right|V_{n,i}^2\left|U_{n,i-1}-U_{n,i-2}\right| \\
\leq\ & \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\left|\mu_s\right|ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 ds\right)\int_{t_{i-2}}^{t_{i-1}}\left|\mu_s-\mu_{s-\Delta_n}\right|ds\left[1+o_p\left(1\right)\right] \\
\leq\ & K\Delta_n^{2+\Gamma/2}\sum_{i=3}^{n}\left|\mu_{t_i}\right|\sigma_{t_i}^2\left[1+o_p\left(1\right)\right] \sim_a K\Delta_n^{1+\Gamma/2}\int_0^T\left|\mu_s\right|\sigma_s^2 ds = O_p\left(T\Delta_n^{1+\Gamma/2}\right),
\end{aligned}$$

using the result in (48) and the boundedness of μ and σ. The same asymptotic order is obtained for all remaining terms in (49) and hence

$$\tilde{Q}_{2,n} = O_p\left(T\Delta_n^{1+\Gamma/2}\right).$$

(3) We now turn to $\tilde{Q}_{3,n}$. Rewrite the first term of $\tilde{Q}_{3,n}$ as

$$\begin{aligned}
& \frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^2 V_{n,i}^2 = \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s dW_s\right)^2\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^2 \\
=\ & \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left[\int_{t_{i-2}}^{t_{i-1}}\sigma_s^2 ds + 2\int_{t_{i-2}}^{t_{i-1}}\left(\int_{t_{i-2}}^{s}\sigma_u dW_u\right)\sigma_s dW_s\right]\left[\int_{t_{i-1}}^{t_i}\sigma_s^2 ds + 2\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)\sigma_s dW_s\right] \\
=\ & \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s^2 ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 ds\right)\left[1+o_p\left(1\right)\right] = \frac{1}{6}\Delta_n\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\left[1+o_p\left(1\right)\right],
\end{aligned}$$

by Ito's lemma and the boundedness of σ. Similarly, we have

$$\begin{aligned}
\frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-2}^2 V_{n,i}^2 &= \frac{1}{6}\Delta_n\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\left[1+o_p\left(1\right)\right], \\
\frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^2 V_{n,i-2}^2 &= \frac{1}{6}\Delta_n\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\left[1+o_p\left(1\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{6\Delta_n}\sum_{i=3}^{n} V_{n,i-1}^4 &= \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s dW_s\right)^4 \\
&= \frac{1}{6\Delta_n}\left[4\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^3\sigma_s dW_s + 6\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^2\sigma_s^2 ds\right] \\
&= \frac{1}{\Delta_n}\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u dW_u\right)^2\sigma_s^2 ds\left[1+o_p\left(1\right)\right] \\
&= \frac{1}{2\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s^2 ds\right)^2\left[1+o_p\left(1\right)\right] = \frac{1}{2}\Delta_n\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\left[1+o_p\left(1\right)\right].
\end{aligned}$$

Consequently,

$$\tilde{Q}_{3,n} = \Delta_n\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\left[1+o_p\left(1\right)\right] \sim_a \int_0^T\sigma_s^4 ds = O_p\left(T\right). \tag{50}$$

In summary, $\tilde{Q}_{1,n} = O_p\left(T\Delta_n^{2+\Gamma/2}\right), \tilde{Q}_{2,n} = O_p\left(T\Delta_n^{1+\Gamma/2}\right)$, and $\tilde{Q}_{3,n} = O_p(T)$ under the double asymptotic setting. Therefore, $\tilde{Q}_{3,n}$ dominates the other two terms and

$$\mathsf{RiceQ} = \tilde{Q}_{3,n}\left[1 + o_p(1)\right] \sim_a \int_0^T \sigma_s^4 ds.$$

The second order term is $\tilde{Q}_{2,n}$ and hence

$$\mathsf{RiceQ} - \tilde{Q}_{3,n} = Q_{2,n}\left[1 + o_p(1)\right] = O_p\left(T\Delta_n^{1+\Gamma/2}\right).$$

■

B.2 Explosive Linear Drift Process

Proof. The RiceQ estimator can be decomposed into three terms:

$$\mathsf{RiceQ}_T = \left(\tilde{Q}_{1,n} + \tilde{Q}_{2,n} + \tilde{Q}_{3,n}\right)\left[1 + o_p(1)\right],$$

where $\tilde{Q}_{1,n}$, $\tilde{Q}_{2,n}$, and $\tilde{Q}_{3,n}$ can be expressed as (47), (49), and (46), respectively. We investigate the consistency of RiceQ_T under the explosive linear drift process. Recall that under the explosive linear drift process

$$U_{n,i} = \int_{t_{i-1}}^{t_i} \mu_s ds = p_0\theta e^{\theta\tau_i}\Delta_n\left[1 + o_p(1)\right] \text{ with } \tau_i \in [t_{i-1}, t_i]. \tag{51}$$

(1) We first investigate the order of $\tilde{Q}_{1,n}$. The first term in (47) has the following properties:

$$\begin{aligned}
\left|\frac{1}{6\Delta_n}\sum_{i=3}^n U_{n,i-1}^3\left(U_{n,i-1} - U_{n,i-2}\right)\right| &= \frac{1}{6\Delta_n}\left|\sum_{i=3}^n\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds\right)^3\left(\int_{t_{i-2}}^{t_{i-1}}\mu_s ds - \int_{t_{i-3}}^{t_{i-2}}\mu_s ds\right)\right| \\
&= \frac{1}{6\Delta_n}\left|\sum_{i=3}^n\left(p_0\theta e^{\theta\tau_{i-1}}\Delta_n\right)^3 p_0\left(e^{\theta\Delta_n} - 1\right)e^{\theta\tau_{i-2}}\Delta_n\right| \\
&= \frac{1}{6}p_0^4\theta^3\left(e^{\theta\Delta_n} - 1\right)\Delta_n^3\sum_{i=3}^n e^{\theta(3\tau_{i-1}+\tau_{i-2})} \\
&\sim_a \frac{1}{6}p_0^4\theta^3\left(e^{\theta\Delta_n} - 1\right)\Delta_n^2\int_0^T e^{4\theta s}ds = O_p\left(e^{4\theta T}\Delta_n^3\right),
\end{aligned}$$

since, from (5),

$$\begin{aligned}
\left|\int_{t_{i-2}}^{t_{i-1}}\mu_s ds - \int_{t_{i-3}}^{t_{i-2}}\mu_s ds\right| &= \int_{t_{i-2}}^{t_i-1}\left|\mu_s - \mu_{s-\Delta_n}\right| ds = \theta p_0\int_{t_{i-2}}^{t_i-1} e^{\theta(s-\Delta_n)}\left(e^{\theta\Delta_n} - 1\right)ds\left[1 + o_p(1)\right] \\
&= p_0 e^{-\theta\Delta_n}\left(e^{\theta\Delta_n} - 1\right)\int_{t_{i-2}}^{t_i-1} e^{\theta s}d(\theta s)\left[1 + o_p(1)\right] \\
&= p_0\left(e^{\theta\Delta_n} - 1\right)\left(e^{\theta t_{i-2}} - e^{\theta t_{i-3}}\right) = p_0\left(e^{\theta\Delta_n} - 1\right)e^{\theta\tau_{i-2}}\Delta_n\left[1 + o_p(1)\right].
\end{aligned} \tag{52}$$

The same asymptotic order is obtained for all remaining terms in $\tilde{Q}_{1,n}$. So

$$\tilde{Q}_{1,n} = O_p\left(e^{4\theta T}\Delta_n^3\right).$$

(2) Next, we investigate the order of $\tilde{Q}_{2,n}$. The first term in (49) is

$$\frac{1}{6\Delta_n}\left|\sum_{i=3}^n U_{n,i-1}\left(U_{n,i-1} - U_{n,i-2}\right)V_{n,i}^2\right| \leq \frac{1}{6\Delta_n}\sum_{i=3}^n\left|U_{n,i-1}\right|V_{n,i}^2\left|U_{n,i-1} - U_{n,i-2}\right|$$

$$\leq \quad K\Delta_n^3 \sum_{i=3}^{n} e^{\theta\left(\tau_{i-1}+\tau'_{i-1}\right)}\sigma_{t_i}^2 \leq K\Delta_n^2 \int_0^T e^{2\theta s}ds = O_p\left(e^{2\theta T}\Delta_n^2\right),$$

using results from (51), the stochastic continuity of σ_s, and, by the mean value theorem,

$$\left|\int_{t_{i-2}}^{t_{i-1}} \mu_s ds - \int_{t_{i-3}}^{t_{i-2}} \mu_s ds\right| = p_0\theta\left|e^{\theta\tau_{i-1}} - e^{\theta\tau_{i-2}}\right|\Delta_n \leq 2p_0\theta^2 e^{\theta\tau'_{i-1}}\Delta_n^2, \tag{53}$$

where $\tau'_{i-1} \in [\tau_{i-2}, \tau_{i-1}]$. The same order is obtained for all remaining terms in $\tilde{Q}_{2,n}$ and therefore

$$\tilde{Q}_{2,n} = O_p\left(e^{2\theta T}\Delta_n^2\right).$$

(3) From (50), the diffusion term

$$\tilde{Q}_{3,n} = \Delta_n \sum_{i=3}^{n} \sigma_{t_{i-1}}^4 \left[1 + o_p(1)\right] \sim_a \int_0^T \sigma_s^4 ds = O_p(T).$$

Consequently, under double asymptotics and the condition that $e^{\theta T}\sqrt{\Delta_n} \to 0$, RiceQ_T is dominated by $\tilde{Q}_{3,n}$ and

$$\mathsf{RiceQ}_T = \int_0^T \sigma_s^4 ds + O_p\left(e^{2\theta T}\Delta_n^2\right).$$

■

B.3 Trending Drift

Proof. Again, the RiceQ estimator can be decomposed into three components:

$$\mathsf{RiceQ}_T = \left(\tilde{Q}_{1,n} + \tilde{Q}_{2,n} + \tilde{Q}_{3,n}\right)\left[1 + o_p(1)\right],$$

where $\tilde{Q}_{1,n}$, $\tilde{Q}_{2,n}$, and $\tilde{Q}_{3,n}$ can be expressed as (47), (49), and (46), respectively. We investigate the asymptotic orders of the three terms. Recall that under the trending drift process,

$$\int_{t_{i-1}}^{t_i} \mu_s ds = \Delta_n\left(\theta_0 + \theta_1 t_i\right)\left[1 + o_p(1)\right]. \tag{54}$$

(1) Using the form of (54), we have

$$\frac{1}{6\Delta_n}\sum_{i=3}^{n} U_{n,i-1}^3\left(U_{n,i-1} - U_{n,i-2}\right) = \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}} \mu_s ds\right)^3\left(\int_{t_{i-2}}^{t_{i-1}} \mu_s ds - \int_{t_{i-3}}^{t_{i-2}} \mu_s ds\right)$$
$$= \quad \frac{1}{6}\theta_1\Delta_n^4\sum_{i=3}^{n}\left(\theta_0 + \theta_1 t_i\right)^3\left[1 + o_p(1)\right] = \frac{1}{24}T^4\Delta_n^3\theta_1^4\left[1 + o_p(1)\right] = O\left(T^4\Delta_n^3\right).$$

All remaining terms in $\tilde{Q}_{1,n}$ have the same asymptotic order and hence

$$\tilde{Q}_{1,n} = O\left(T^4\Delta_n^3\right).$$

(2) The first term in (49) has the following properties

$$\frac{1}{6\Delta_n}\left|\sum_{i=3}^{n} U_{n,i-1}\left(U_{n,i-1} - U_{n,i-2}\right)V_{n,i}^2\right| = \frac{1}{6\Delta_n}\left|\sum_{i=3}^{n} U_{n,i-1}V_{n,i}^2\left(U_{n,i-1} - U_{n,i-2}\right)\right|$$

$$
\begin{aligned}
&= \frac{1}{6}\theta_1 \Delta_n^2 \sum_{i=3}^{n} (\theta_0 + \theta_1 t_{i-1}) \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \right)^2 \\
&= \frac{1}{6}\theta_1 \Delta_n^3 \sum_{i=3}^{n} (\theta_0 + \theta_1 t_{i-1}) \sigma_{t_{i-1}}^2 \left[1 + o_p(1)\right] \\
&\sim_a \frac{1}{6}\theta_1 \Delta_n^2 \int_0^T (\theta_0 + \theta_1 s) \sigma_s^2 ds = O_p\left(T^2 \Delta_n^2\right).
\end{aligned}
$$

The same asymptotic order is obtained for all other terms in (49) and hence

$$
\tilde{Q}_{2,n} = O_p\left(T^2 \Delta_n\right).
$$

(3) From (50), the diffusion term

$$
\tilde{Q}_{3,n} = \Delta_n \sum_{i=3}^{n} \sigma_{t_{i-1}}^4 \left[1 + o_p(1)\right] \sim_a \int_0^T \sigma_s^4 ds = O_p(T).
$$

In summary, under the double asymptotic, we have $\tilde{Q}_{1,n} = O\left(T^4 \Delta_n^3\right)$, $\tilde{Q}_{2,n} = O_p\left(T^2 \Delta_n\right)$ and $\tilde{Q}_{3,n} = O_p(T)$. When $T\Delta_n \to 0$, RiceQ_T is dominated by $\tilde{Q}_{3,n}$ and

$$
\mathsf{RiceQ}_T = \tilde{Q}_{3,n} + O_p\left(T^2 \Delta_n\right) \sim_a \int_0^T \sigma_s^4 ds.
$$

■

B.4 Drift Burst

Proof. We derive the consistency of RiceQ_T under the drift burst process. Let $U_{n,i}^{\dagger} = \int_{t_{i-1}}^{t_i} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds$ and $V_{n,i}^{\dagger} = \int_{t_{i-1}}^{t_i} \sigma_s \left(1 - \frac{s}{T}\right)^{-\beta} dW_s$. Under the model specification,

$$
\mathsf{RiceQ}_T = \left(Q_{1,n}^{\dagger} + Q_{2,n}^{\dagger} + Q_{3,n}^{\dagger}\right) \left[1 + o_p(1)\right]
$$

Here, $Q_{1,n}^{\dagger}$, $Q_{2,n}^{\dagger}$, and $Q_{3,n}^{\dagger}$ take the same form as $Q_{1,n}$, $Q_{2,n}$, and $Q_{3,n}$ respectively, with U replaced by $U^{\dagger}$ and V replaced by $V^{\dagger}$.

Under the drift bursting process, the quantity

$$
\begin{aligned}
& \int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds - \int_{t_{i-3}}^{t_{i-2}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \\
= & \int_{t_{i-2}}^{t_{i-1}} \left[\mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} - \mu_{s-\Delta_n} \left(1 - \frac{s - \Delta_n}{T}\right)^{-\alpha} \right] ds \\
= & \int_{t_{i-2}}^{t_{i-1}} \left(\mu_s - \mu_{s-\Delta_n}\right) \left(1 - \frac{s}{T}\right)^{-\alpha} ds + \int_{t_{i-2}}^{t_{i-1}} \mu_{s-\Delta_n} \left[\left(1 - \frac{s}{T}\right)^{-\alpha} - \left(1 - \frac{s - \Delta_n}{T}\right)^{-\alpha} \right] ds \\
= & \int_{t_{i-2}}^{t_{i-1}} \mu_{s-\Delta_n} \left[\left(1 - \frac{s}{T}\right)^{-\alpha} - \left(1 - \frac{s - \Delta_n}{T}\right)^{-\alpha} \right] ds \left[1 + o_p(1)\right] \\
\leq & K T^{\alpha} \Delta_n^{1-\alpha} \tilde{\xi}_j^{-\alpha-1} = O_p\left(T^{\alpha} \Delta_n^{1-\alpha}\right),
\end{aligned}
\tag{55}
$$

where $\tilde{\xi}_j \in [\xi_j, \xi_{j+1}]$ with $\xi_j \in [j, j+1]$ and $j = n - i + 1$, since under the Hölder continuity assumption of μ_s,

$$\int_{t_{i-1}}^{t_i} |\mu_s - \mu_{s-\Delta_n}| \left(1 - \frac{s}{T}\right)^{-\alpha} ds \leq K \Delta_n^{\Gamma/2} \int_{t_{i-1}}^{t_i} \left(1 - \frac{s}{T}\right)^{-\alpha} ds = O_p\left(T^\alpha \Delta_n^{1-\alpha+\Gamma/2}\right), \tag{56}$$

and by the boundedness of μ_s and the notation $j = n - i + 1$,

$$\begin{aligned}
& \left| \int_{t_{i-2}}^{t_{i-1}} \mu_{s-\Delta_n} \left[\left(1 - \frac{s}{T}\right)^{-\alpha} - \left(1 - \frac{s - \Delta_n}{T}\right)^{-\alpha} \right] ds \right| \\
\leq \; & K T^\alpha \Delta_n^{-\alpha} \left[\int_{t_{i-2}}^{t_{i-1}} \left(n - \frac{s}{\Delta_n}\right)^{-\alpha} ds - \int_{t_{i-2}}^{t_{i-1}} \left(n - \frac{s}{\Delta_n} + 1\right)^{-\alpha} ds \right] \\
= \; & \frac{K}{1-\alpha} T^\alpha \Delta_n^{1-\alpha} \left[j^{1-\alpha} - 2(j+1)^{1-\alpha} + (j+2)^{1-\alpha} \right] \\
\leq \; & K T^\alpha \Delta_n^{1-\alpha} \tilde{\xi}_j^{-\alpha-1} = O_p\left(T^\alpha \Delta_n^{1-\alpha}\right).
\end{aligned}$$

Note that by mean value theorem, $j^{1-\alpha} - (j+1)^{1-\alpha} = -(1-\alpha)\xi_j^{-\alpha}$ and

$$j^{1-\alpha} - 2(j+1)^{1-\alpha} + (j+2)^{1-\alpha} = -(1-\alpha)\left(\xi_j^{-\alpha} - \xi_{j+1}^{-\alpha}\right) = \alpha(1-\alpha)\tilde{\xi}_j^{-\alpha-1},$$

where $j = n - i + 1$.

(1) We first investigate the asymptotic order of $Q_{1,n}^\dagger$. Rewrite $Q_{1,n}^\dagger$ as in (47), where we replace U with $U^\dagger$. The first term has the following asymptotic properties:

$$\begin{aligned}
& \left| \frac{1}{6\Delta_n} \sum_{i=3}^n U_{n,i-1}^{\dagger 4} \left(U_{n,i-1}^\dagger - U_{n,i-2}^\dagger\right) \right| \\
= \; & \left| \frac{1}{6\Delta_n} \sum_{i=3}^n \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right)^3 \left(\int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds - \int_{t_{i-3}}^{t_{i-2}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right) \right| \\
\leq \; & K T^{4\alpha} \Delta_n^{3-4\alpha} \sum_{j=1}^{n-2} \xi_j^{-3\alpha} \tilde{\xi}_j^{-\alpha-1} \left[1 + o_p(1)\right] \sim_a K T^{4\alpha} \Delta_n^{3-4\alpha} \sum_{j=1}^{n-2} j^{-4\alpha-1}
\end{aligned}$$

using results in Lemma A.1 and (55). When $\alpha > 0$, $S_n = \sum_{j=1}^{n-2} j^{-4\alpha-1}$ converges and hence the order of the differenced quantity is $O_p\left(T^{4\alpha}\Delta_n^{3-4\alpha}\right)$ under the double asymptotic setting. The same asymptotic order is obtained for all remaining terms in $Q_{1,n}^\dagger$. Therefore,

$$Q_{1,n}^\dagger = O_p\left(T^{4\alpha}\Delta_n^{3-4\alpha}\right).$$

(2) Rewrite $Q_{2,n}^\dagger$ as in (49) where we replace U with $U^\dagger$ and V with $V^\dagger$. The first term is

$$\begin{aligned}
& \frac{1}{6\Delta_n} \left| \sum_{i=3}^n U_{n,i-1}^\dagger \left(U_{n,i-1}^\dagger - U_{n,i-2}^\dagger\right) V_{n,i}^{\dagger 2} \right| \leq \frac{1}{6\Delta_n} \sum_{i=3}^n \left|U_{n,i-1}^\dagger\right| V_{n,i}^{\dagger 2} \left|U_{n,i-1}^\dagger - U_{n,i-2}^\dagger\right| \\
\leq \; & \frac{1}{6\Delta_n} \sum_{i=3}^n \left| \int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right| \left| \int_{t_{i-1}}^{t_i} \sigma_s^2 \left(1 - \frac{s}{T}\right)^{-2\beta} ds \right| \\
& \left| \int_{t_{i-2}}^{t_{i-1}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds - \int_{t_{i-3}}^{t_{i-2}} \mu_s \left(1 - \frac{s}{T}\right)^{-\alpha} ds \right| \left[1 + o_p(1)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq KT^{2\alpha+2\beta}\Delta_n^{2-2\alpha-2\beta}\sum_{j=1}^{n-2}\xi_j^{-\alpha}\xi_{j-1}'^{-2\beta}\tilde{\xi}_j^{-\alpha-1}\left[1+o_p\left(1\right)\right] \\
&= KT^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}\sum_{j=1}^{n-2}j^{-2(\alpha+\beta)-1}\left[1+o_p\left(1\right)\right]=O_p\left(T^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}\right),
\end{aligned}
$$

using results in Lemma A.1 and (55) and the convergence of $\sum_{j=1}^{n-2}j^{-2(\alpha+\beta)-1}$. All remaining terms in (49) have the same order. Therefore,

$$
Q_{2,n}^{\dagger}=O_p\left(T^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}\right).
$$

(3) For $Q_{3,n}^{\dagger}$, the first term

$$
\begin{aligned}
&\frac{1}{6\Delta_n}\sum_{i=3}^{n}V_{n,i-1}^{\dagger 2}V_{n,i}^{\dagger 2} \\
=\ &\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right)^2\left(\int_{t_{i-1}}^{t_i}\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right)^2 \\
=\ &\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left[\int_{t_{i-2}}^{t_{i-1}}\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds+2\int_{t_{i-2}}^{t_{i-1}}\left(\int_{t_{i-2}}^{s}\sigma_u\left(1-\frac{u}{T}\right)^{-\beta}dW_u\right)\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right] \\
&\left[\int_{t_{i-1}}^{t_i}\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds+2\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u\left(1-\frac{u}{T}\right)^{-\beta}dW_u\right)\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right] \\
=\ &\frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds\right)\left(\int_{t_{i-1}}^{t_i}\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds\right)\left[1+o_p\left(1\right)\right] \\
=\ &\frac{1}{6}T^{4\beta}\Delta_n^{1-4\beta}\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\xi_{n-i+1}'^{-2\beta}\xi_{n-i}'^{-2\beta}\left[1+o_p\left(1\right)\right]=\frac{1}{6}T^{4\beta}\Delta_n^{1-4\beta}\sum_{j=1}^{n-2}\sigma_{t_{n-j}}^4j^{-4\beta}\left[1+o_p\left(1\right)\right].
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
\frac{1}{6\Delta_n}\sum_{i=3}^{n}V_{n,i-2}^{\dagger 2}V_{n,i}^{\dagger 2} &= \frac{1}{6}T^{4\beta}\Delta_n^{1-4\beta}\sum_{j=1}^{n-2}\sigma_{t_{n-j}}^4j^{-4\beta}\left[1+o_p\left(1\right)\right], \\
\frac{1}{6\Delta_n}\sum_{i=3}^{n}V_{n,i-1}^{\dagger 2}V_{n,i-2}^{\dagger 2} &= \frac{1}{6}T^{4\beta}\Delta_n^{1-4\beta}\sum_{j=1}^{n-2}\sigma_{t_{n-j}}^4j^{-4\beta}\left[1+o_p\left(1\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{6\Delta_n}\sum_{i=3}^{n}V_{n,i-1}^{\dagger 4} &= \frac{1}{6\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right)^4 \\
&= \frac{1}{6\Delta_n}\left[4\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u\left(1-\frac{u}{T}\right)^{-\beta}dW_u\right)^3\sigma_s\left(1-\frac{s}{T}\right)^{-\beta}dW_s\right. \\
&\quad\left.+6\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u\left(1-\frac{u}{T}\right)^{-\beta}dW_u\right)^2\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds\right]\left[1+o_p\left(1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Delta_n}\sum_{i=3}^{n}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\sigma_u\left(1-\frac{u}{T}\right)^{-\beta}dW_u\right)^2\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds\left[1+o_p\left(1\right)\right] \\
&= \frac{1}{2\Delta_n}\sum_{i=3}^{n}\left(\int_{t_{i-1}}^{t_i}\sigma_s^2\left(1-\frac{s}{T}\right)^{-2\beta}ds\right)^2\left[1+o_p\left(1\right)\right] \\
&= \frac{1}{2}T^{4\beta}\Delta_n^{1-4\beta}\sum_{i=3}^{n}\sigma_{t_{i-1}}^4\xi_{n-i}'^{-4\beta}\left[1+o_p\left(1\right)\right]=\frac{1}{2}T^{4\beta}\Delta_n^{1-4\beta}\sum_{j=1}^{n-2}\sigma_{t_{j-j}}^4 j^{-4\beta}\left[1+o_p\left(1\right)\right].
\end{aligned}
$$

It follows that

$$
Q_{3,n}^{\dagger}=T^{4\beta}\Delta_n^{1-4\beta}\sum_{j=1}^{n-2}\sigma_{t_{j-j}}^4 j^{-4\beta}\left[1+o_p\left(1\right)\right]\sim_a\int_0^T\sigma_s^4\left(1-\frac{s}{T}\right)^{-4\beta}ds,
$$

since $\beta<1/4$. The quantity has an order of $O_p\left(T\right)$ under the double asymptotic setting.

In summary, under the double asymptotic scheme, $Q_{1,n}^{\dagger}=O_p\left(T^{4\alpha}\Delta_n^{3-4\alpha}\right)$, $Q_{2,n}^{\dagger}=O_p\left(T^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}\right)$, and $Q_{3,n}^{\dagger}=O_p\left(T\right)$. The conditions required for the diffusion term $Q_{3,n}^{\dagger}$ to dominate $Q_{1,n}^{\dagger}$ and $Q_{2,n}^{\dagger}$ are

$$
\begin{aligned}
\frac{Q_{3,n}^{\dagger}}{Q_{2,n}^{\dagger}} &: \frac{T}{T^{2(\alpha+\beta)}\Delta_n^{2-2(\alpha+\beta)}}=\left[T^{1/2-(\alpha+\beta)}\Delta_n^{\alpha+\beta-1}\right]^2\to\infty, \\
\frac{Q_{3,n}^{\dagger}}{Q_{1,n}^{\dagger}} &: \frac{T}{T^{4\alpha}\Delta_n^{3-4\alpha}}=T^{1-4\alpha}\Delta_n^{4\alpha-3}\to\infty.
\end{aligned}
$$

When $\alpha\leq 1/4$, both conditions are satisfied if $T\to\infty$ and $\Delta_n\to 0$. When $\alpha>1/4$ and $\alpha+\beta\leq 1/2$, the first condition is satisfied under the double asymptotic setting. Let $T=c\Delta_n^{-\psi}$ and rewrite the second condition as

$$
-\psi\left(1-4\alpha\right)-3+4\alpha<0\Longrightarrow\psi<\frac{3-4\alpha}{4\alpha-1}.
$$

When $\alpha>1/4$ and $\alpha+\beta>1/2$, the first condition can be rewritten as

$$
-\psi\left[1/2-(\alpha+\beta)\right]+\alpha+\beta-1<0\Longrightarrow\psi<\frac{1-(\alpha+\beta)}{(\alpha+\beta)-1/2},
$$

and the second condition implies $\psi<\frac{3-4\alpha}{4\alpha-1}$. So, the conditions require

$$
\psi<\min\left\{\frac{3-4\alpha}{4\alpha-1},\frac{1-(\alpha+\beta)}{(\alpha+\beta)-1/2}\right\}.
$$

Therefore, we have

$$
\mathsf{RiceQ}=Q_{3,n}^{\dagger}\left[1+o_p\left(1\right)\right]\sim_a\int_0^T\sigma_s^4\left(1-\frac{s}{T}\right)^{-4\beta}ds,
$$

if the following additional conditions are satisfied under the double asymptotic setting:

$$
\begin{cases}
T^{1-4\alpha}\Delta_n^{4\alpha-3}\to\infty & \text{if } \alpha>1/4 \text{ and } \alpha+\beta\leq 1/2 \\
T^{1-4\alpha}\Delta_n^{4\alpha-3}\to\infty \text{ and } T^{1/2-(\alpha+\beta)}\Delta_n^{\alpha+\beta-1}\to\infty & \text{if } \alpha>1/4 \text{ and } \alpha+\beta>1/2
\end{cases}.
$$

or

$$
\begin{cases}
\psi<\frac{3-4\alpha}{4\alpha-1} & \text{if } \alpha>1/4 \text{ and } \alpha+\beta\leq 1/2 \\
\psi<\min\left\{\frac{3-4\alpha}{4\alpha-1},\frac{1-(\alpha+\beta)}{(\alpha+\beta)-1/2}\right\} & \text{if } \alpha>1/4 \text{ and } \alpha+\beta>1/2
\end{cases}.
$$

■

C Jump Robustness

In the presence of jumps the logarithmic price of a financial asset is commonly modeled as a jump diffusion process expressed as

$$p_t^J = p_t + J_t, \tag{57}$$

where p_t is specified in (1) and J_t denotes a jump component of the following form:

$$J_t = \int_0^t \int_{\mathbb{R}} g(s,x)\, \upsilon(ds,dx)$$

where $g : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is a predictable mapping capturing the impact of jumps, and υ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\upsilon(ds,dx) = ds \otimes F(dx)$.

Assumption C.1 *Suppose that there is a localizing sequence of stopping times (T_n) and for each n, a deterministic nonnegative function Γ_n defined on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} |\Gamma_n(x)|^r F(dx) < \infty$, for some $r \in [0,1)$, and such that $|g(s,x)| \wedge 1 \leq \Gamma_n(x)$ for all (ω, s, x) with $s \leq T_n(w)$.*

Denote the jump contaminated return by $r_{n,i}^J = r_{n,i} + \Delta_n^i J$ with $\Delta_n^i J = J_{t_i} - J_{t_{i-1}}$. Under the jump diffusion process, the realized drift estimator has the following asymptotic properties:

$$\begin{aligned}
\frac{1}{\sqrt{n}} \mathsf{RD}_T &= \frac{1}{\sqrt{n}} \left(\frac{1}{\Delta_n} \sum_{i=2}^n r_{n,i}^J r_{n,i-1}^J \right) \\
&= \left[n^{-1/2} D_{3,n} + \frac{1}{T} \left(\sqrt{n} \sum_{i=2}^n \Delta_n^i J \right) \right] [1 + o_p(1)] \\
&\to^L \mathcal{MN}\left(0, \omega_{\sigma^4} + \frac{1}{T} \sum_{s \in [0,T]} r_s^2 \left(\sigma_{s^-}^2 + \sigma_s^2 \right) \right),
\end{aligned}$$

where σ_{s^-} and σ_s are the spot variance to the left and right of s, respectively. The asymptotic dominance and convergence follow directly from results in Lemma 3.1 and Jacod and Protter (2011b, Theorem 5.4.6). Rewrite

$$\sqrt{\Delta_n}\, \mathsf{RD}_T \sim_a \mathcal{MN}\left(0, \int_0^T \sigma_s^4 ds + \sum_{s \in [0,T]} r_s^2 \left(\sigma_{s^-}^2 + \sigma_s^2 \right) \right). \tag{58}$$

The realized drift estimator remains unbiased but with inflated variance in the presence of jumps. The variance induced by jumps (i.e., $\sum_{s \in [0,T]} r_s^2 \left(\sigma_{s^-}^2 + \sigma_s^2 \right)$) can be estimated by:

$$V_T^J = \sum_{i=2}^n (r_{n,i} - r_{n,i-1})^2 1_{\{|r_{n,i} - r_{n,i-1}| > \varsigma \Delta_n^{\varpi}\}} \left(\hat{c}_{i,-}^n + \hat{c}_i^n \right),$$

where $\hat{c}_{i,-}^n$ and $\hat{c}_i^n$ are the spot variance estimators for $\sigma_{s^-}^2$ and σ_s^2, respectively, given by

$$\begin{aligned}
\hat{c}_{i,-}^n &= \frac{1}{\Delta_n k_n} \frac{1}{2} \sum_{j=1}^{k_n} (r_{n,i-j-1} - r_{n,i-j-2})^2 1_{\{|r_{n,i-j-1} - r_{n,i-j-2}| \leq \varsigma \Delta_n^{\varpi}\}}, \\
\hat{c}_i^n &= \frac{1}{\Delta_n k_n} \frac{1}{2} \sum_{j=1}^{k_n} (r_{n,i+j+1} - r_{n,i+j})^2 1_{\{|r_{n,i+j+1} - r_{n,i+j}| \leq \varsigma \Delta_n^{\varpi}\}},
\end{aligned}$$

for some sequence $k_n \to \infty$ with $\Delta_n k_n \to 0$. See Andersen et al. (2023) for more details.

Andersen et al. (2023) introduced a jump robust test for gradual jumps and persistent noise, denoted as $S_{3,T}$. It is defined by:

$$S_{3,T} = \frac{\mathsf{RD}_T}{\sqrt{\mathsf{RQ}_T^J / \Delta_n}}$$

where $\mathsf{DVQ}_T^J := \mathsf{DVQ}_T + V_T^J \wedge (\mathsf{DVQ}_T \log n)$ with $\wedge$ indicating infimum and

$$\mathsf{DVQ}_T = \frac{1}{12\Delta_n} \sum_{i=2}^{n} (r_{n,i} - r_{n,i-1})^4 \, 1_{\{|r_{n,i} - r_{n,i-1}| \leq \varsigma \Delta_n^{\varpi}\}}.$$

The test statistic is shown to converge to the standard normal distribution under the zero drift jump diffusion process. We consider a slightly modified version of their test, denoted as $S_{4,T}$ and defined by:

$$S_{4,T} = \frac{\mathsf{RD}_T}{\sqrt{\mathsf{RiceQ}_T^J / \Delta_n}},$$

where $\mathsf{RiceQ}_T^J := \mathsf{RiceQ}_T^* + V_T^J \wedge (\mathsf{RiceQ}_T^* \log n)$ with

$$\mathsf{RiceQ}_T^* = \frac{1}{6\Delta_n} \sum_{i=3}^{n} (r_{n,i} - r_{n,i-1})^2 (r_{n,i-1} - r_{n,i-2})^2 \, 1_{\{|r_{n,i} - r_{n,i-1}| \leq \varsigma \Delta_n^{\varpi}\}} 1_{\{|r_{n,i-1} - r_{n,i-2}| \leq \varsigma \Delta_n^{\varpi}\}}.$$

For computation, we set $\varsigma = C_\varsigma \sqrt{\mathsf{MedRV}}$, $\varpi = 1/2$ and $C_\varsigma = 4$ as in Andersen et al. (2021). The MedRV is proposed by Andersen et al. (2012) and defined as:

$$\mathsf{MedRV} = \frac{\pi}{6 - 4\sqrt{3} + \pi} \left(\frac{n}{n+2} \right) \sum_{i=3}^{n-1} \mathrm{med} \left(|r_{n,i-2}|, |r_{n,i-1}|, |r_{n,i}| \right)^2.$$

C.1 Simulation Studies

Here, we assume that J_t follows a compound Poisson process with intensity λ and jump size δ_t following a zero mean Gaussian process $N(0, \sigma_J^2)$. As in Andersen et al. (2023), we set $\lambda = 1/5$ which corresponds to one jump per week on average and $\sigma_J = 0.9\%$. The data generating process and parameter settings for p_t are identical to those in Section 6. We first investigate the estimation accuracy of RD_T for ID_T in the presence of jumps, with a nominal size of 10%.

Table 3 presents the mean and standard deviation of $\frac{1}{T}(\mathsf{RD}_T - \mathsf{ID}_T) \times 10^3$ across different drift scenarios in the presence of jumps. Comparing with Table 1, despite the noticeable increases in standard deviation there are no significant changes in bias. This is consistent with the asymptotic results in equation (58). The influence of jumps is more pronounced if T is larger as it contains more jumps on average.

We proceed by comparing the performance of $S_{1,T}$ and $S_{2,T}$, as defined in equation (23), with their respective jump-robust counterparts $S_{3,T}$ and $S_{4,T}$. The presence of jumps introduces a downward bias in size under the null hypothesis of no drift, consequently leading to reduced power under the alternative hypothesis. This effect becomes more pronounced as the sample size T increases. However, the issue of size distortion is less severe in the jump-robust tests compared to their original counterparts. For instance, when the time span is two weeks (i.e., $T = 10$), the size of the jump-robust test $S_{4,T}$ is 6%, whereas the original versions exhibit a size of 5%. This slight enhancement carries significant implications for power. For example, under the scenario of explosive linear drift with parameters $\theta = 0.005$ and $T = 10$, the power

Table 3: The mean and standard deviation (in parentheses) of $\frac{1}{T}(\mathsf{RD}-\mathsf{ID})\times 10^3$ under various parameter settings in the presence of jumps. The number of replications is $2,000$.

Coeff	$T=1$	$T=5$	$T=10$	$T=20$
Zero Drift				
$\mu=0$	0.04 (1.07)	-0.01 (0.46)	-0.01 (0.35)	0.01 (0.23)
Constant Drift				
$\mu=0.01$	-0.05 (1.05)	-0.02 (0.52)	-0.01 (0.36)	-0.01 (0.24)
$\mu=0.03$	-0.04 (1.25)	-0.03 (0.53)	-0.03 (0.39)	-0.04 (0.29)
$\mu=0.05$	-0.13 (1.48)	-0.08 (0.65)	-0.09 (0.51)	-0.11 (0.34)
Stationary Linear Drift				
$\theta=-0.001$	-0.05 (1.06)	-0.00 (0.52)	-0.01 (0.34)	-0.00 (0.25)
$\theta=-0.003$	-0.04 (1.09)	-0.01 (0.54)	-0.03 (0.36)	-0.02 (0.25)
$\theta=-0.005$	-0.06 (1.26)	-0.04 (0.57)	-0.05 (0.41)	-0.04 (0.30)
Explosive Linear Drift				
$\theta=0.001$	-0.02 (1.12)	0.01 (0.52)	0.01 (0.35)	-0.00 (0.25)
$\theta=0.003$	-0.10 (1.19)	-0.03 (0.50)	-0.01 (0.36)	-0.02 (0.28)
$\theta=0.005$	-0.08 (1.25)	-0.06 (0.58)	-0.06 (0.42)	-0.06 (0.29)
Trending Drift				
$\theta_1=-0.01$	-0.01 (1.13)	0.00 (0.51)	0.01 (0.36)	0.00 (0.26)
$\theta_1=-0.03$	-0.02 (1.13)	0.00 (0.48)	-0.01 (0.37)	-0.01 (0.25)
$\theta_1=-0.05$	-0.05 (1.21)	-0.04 (0.50)	-0.03 (0.38)	-0.03 (0.28)
Bursting Drift: $\mu=0.02$				
$\alpha=0.1$	-0.03 (1.11)	-0.01 (0.50)	-0.02 (0.38)	-0.02 (0.27)
$\alpha=0.2$	-0.09 (1.16)	-0.04 (0.51)	-0.04 (0.38)	-0.05 (0.27)
$\alpha=0.3$	-0.21 (1.30)	-0.13 (0.54)	-0.10 (0.40)	-0.09 (0.27)

of $S_{3,T}$ is 20% higher than that of $S_{1,T}$, and the power of $S_{4,T}$ is 10% higher than that of $S_{2,T}$. Notably, the performance of the two jump-robust tests appears indistinguishable.

Overall, simulation results suggest that drift tests with large T may exhibit conservatism when the sample period includes non-negligible jumps. Despite this, it is evident that all four tests, especially the jump-robust tests, still exhibit good power across all five categories of drifts.

Table 4: Sizes and powers of the drift tests under various parameter settings in the presence of jumps. The nominal size of the tests is 10%. The number of replications is $2,000$.

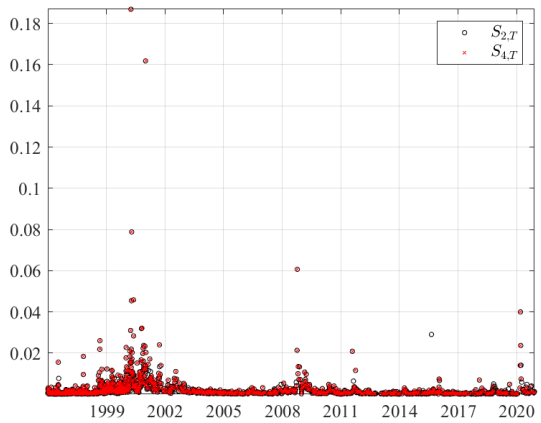
Coeff	$T=1$				$T=5$				$T=10$				$T=20$			
	$S_{1,T}$	$S_{2,T}$	$S_{3,T}$	$S_{4,T}$	$S_{1,T}$	$S_{2,T}$	$S_{3,T}$	$S_{4,T}$	$S_{1,T}$	$S_{2,T}$	$S_{3,T}$	$S_{4,T}$	$S_{1,T}$	$S_{2,T}$	$S_{3,T}$	$S_{4,T}$
Zero Drift																
$\mu=0$	0.09	0.11	0.12	0.12	0.05	0.07	0.07	0.07	0.04	0.05	0.05	0.06	0.02	0.03	0.03	0.03
Constant Drift																
$\mu=0.01$	0.12	0.16	0.16	0.17	0.10	0.14	0.16	0.17	0.08	0.11	0.14	0.15	0.04	0.06	0.10	0.10
$\mu=0.03$	0.41	0.48	0.51	0.52	0.49	0.56	0.66	0.67	0.49	0.60	0.73	0.73	0.55	0.70	0.80	0.80
$\mu=0.05$	0.68	0.73	0.77	0.78	0.80	0.87	0.94	0.94	0.86	0.93	0.97	0.97	0.96	0.98	0.99	0.99
Stationary Linear Drift																
$\theta=-0.001$	0.08	0.12	0.12	0.12	0.07	0.10	0.11	0.11	0.04	0.05	0.07	0.07	0.03	0.04	0.05	0.05
$\theta=-0.003$	0.25	0.31	0.32	0.32	0.29	0.34	0.42	0.42	0.24	0.31	0.43	0.43	0.20	0.31	0.41	0.42
$\theta=-0.005$	0.49	0.56	0.57	0.58	0.59	0.66	0.76	0.76	0.57	0.67	0.80	0.81	0.63	0.76	0.86	0.86
Explosive Linear Drift																
$\theta=0.001$	0.11	0.14	0.14	0.15	0.07	0.09	0.11	0.11	0.06	0.08	0.09	0.09	0.03	0.05	0.05	0.06
$\theta=0.003$	0.25	0.30	0.31	0.32	0.30	0.36	0.44	0.44	0.32	0.40	0.51	0.52	0.30	0.41	0.54	0.55
$\theta=0.005$	0.48	0.56	0.58	0.58	0.57	0.66	0.77	0.77	0.65	0.74	0.85	0.85	0.80	0.90	0.94	0.94
Trending Drift																
$\theta_1=-0.01$	0.10	0.12	0.12	0.13	0.07	0.08	0.10	0.10	0.04	0.05	0.06	0.06	0.02	0.04	0.04	0.04
$\theta_1=-0.03$	0.20	0.26	0.26	0.27	0.23	0.28	0.34	0.35	0.20	0.25	0.33	0.33	0.16	0.24	0.33	0.33
$\theta_1=-0.05$	0.38	0.46	0.47	0.47	0.49	0.56	0.65	0.66	0.49	0.59	0.71	0.72	0.50	0.64	0.77	0.77
Bursting Drift: $\mu=0.02$																
$\alpha=0.1$	0.28	0.34	0.36	0.36	0.34	0.40	0.48	0.48	0.29	0.38	0.49	0.50	0.29	0.42	0.55	0.55
$\alpha=0.2$	0.33	0.41	0.41	0.42	0.38	0.46	0.56	0.56	0.40	0.48	0.61	0.62	0.41	0.55	0.67	0.68
$\alpha=0.3$	0.37	0.45	0.47	0.47	0.49	0.57	0.67	0.67	0.51	0.61	0.75	0.75	0.56	0.72	0.82	0.82

C.2 Empirical Application

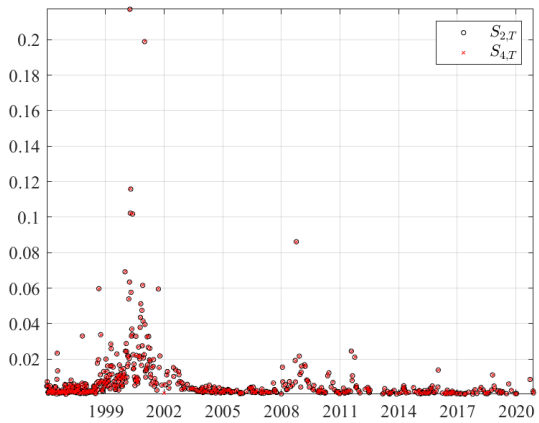
Here, we revisit the Nasdaq application and compare the results obtained from $S_{2,T}$ and its jump robust version $S_{4,T}$. Figure 8 presents the significant realized drift estimates from these two tests. The black circles represent results from $S_{2,T}$ (based on RiceQ_T), while the red crosses depict those from $S_{4,T}$ (based on RiceQ_T^J). No visible differences are observed in the results, suggesting that jumps do not play a significant role in the Nasdaq composite index. However, jumps may have a more prominent impact on individual stocks, the analysis of which would require a more refined version of the drift test. We leave this aspect for future investigation.

Figure 8: Significant realised drift estimates: Nasdaq composite index 1996 to 2020. The black circles are from S_{2T}, while the red crosses are from the jump robust test statistic $S_{4,T}$.

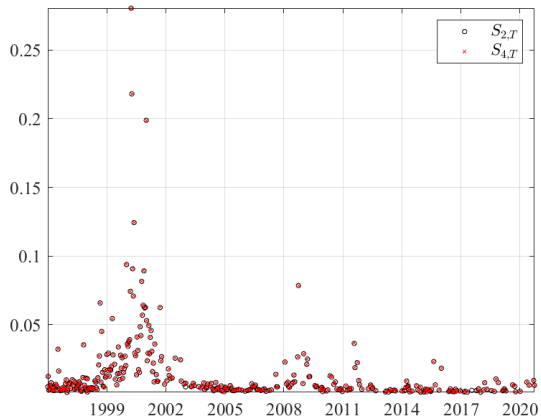
(a) Daily



(b) Weekly



(c) Fornightly



(d) Monthly