A simple parsimonious framework for extracting and modelling the term structure of interest rates*

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Abstract

This paper introduces a novel methodology for the extraction and modelling of the unobserved term structure of interest rates which incorporates in a single inferential framework both cross-sectional and time-series information from observed bond prices. In doing so, the paper introduces both a parametric and a semi-parametric dynamic model for the term structure which can be extended to capture also features such as heteroscedasticity in both the time and cross-section dimension, as well as the zero-lowerbound constraint. The models provide a coherent description of the term structure and outperform current term structure extraction methods in fitting the observed bond prices surface. The models also outperform other existing dynamic term structure models in forecasting observed bond prices at one, six and twelve months horizons. Moreover, the study highlights a strong sensitivity of the forecasting errors of the existing dynamic models to the choice of the term structure extraction method used to construct the samples for the parameters' estimation.

Keywords: Term Structure Modelling, Dynamic Factor Models, Structural Time Series, Dynamic Splines, Kalman Filter, Score-Driven Models, Heteroscedasticity, Zero-Lowerbound
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1. Introduction

The estimation and forecasting of the term structure of interest rates has always been an important topic of research due to its relevance in many fields such as, among others, risk management and monetary policy. However, since interest rates are not usually directly observable, in practice is possible to infer information on the behaviour of the term structure only from the market price of traded bonds.

By the non-arbitrage theory of asset prices the price of a coupon bearing bond traded on the market at any point in time is uniquely defined by three elements: the time to maturity of the bond, its cashflows and the current term structure of interest rates[1]. Given that the first two are predetermined at issuance and are fixed for the entire "*lifespan*" of a bond, in absence of measurement errors the fluctuations of the bond price over time should only depend on the fluctuations of the underlying term structure of interest rates. However, since the non-arbitrage relation behind bond prices is rarely verified empirically, and there are few numbers of traded bonds on the market at longer maturities[2], the estimation and inference about the yield curve poses several issues.

There is a vast literature focused on statistical techniques aimed at extracting the term structure from observed bond prices. At the same time, there is also a vast literature on the modelling of the extracted term structure data. However, the state of art of the dynamic term structure models are focused only on modelling time series of previously extracted term structure curves, neglecting their accuracy in forecasting the actual observed bond prices. The fact that these models are estimated on previously synthetic extracted term structure data data is not usually acknowledged.

The aim of this paper is to propose a novel single framework for the modelling of the term structure of interest rates which incorporates both the cross-sectional and the time-series information in bond prices data. Then its accuracy is compared against other stylized

[1]Given the price of a bond p_t with maturity τ_j and coupon c_j with payment dates $\tau_1 < \tau_2 < \ldots < \tau_j$, if the term structure of interest rates at time t is defined as $r_t(\tau_1), r_t(\tau_2), \ldots, r_t(\tau_j)$ then its price will be defined as $p_t(\tau_j) = c_j\, d_t(\tau_1) + c_j\, d_t(\tau_2) + \ldots + c_j\, d_t(\tau_{j-1}) + (1 + c_j)\, d_t(\tau_j)$, where $d_t(\tau) = e^{-\tau\, r_t(\tau)}$ are the discount factors, or discount rates, from the discount function $\tau \to d_t$ in discrete and continuous compounding. If the bond is a zero coupon bond its price will be $p_t(\tau_j) = d_t(\tau_j)$. For a detailed discussion of this non-arbitrage price relation and the compounding conventions see Hull (2014).

[2]For these reasons, the term structure of interest rates is usually not uniquely identified in bond prices without imposing further assumptions or using instrumental variables, Campbell et al. (1997)

methodologies on the fitting and forecasting of the observed bond prices series. To do so, treating the term structure as a latent unobserved process, we propose two simple filtering methods to estimate and model the entire unobserved signal term structure over time. Both models provide a coherent representation of the term structure. The second can be extended to include also additional features, such as heteroscedastiticity in the time dimension and a zero-lowerbound. The models also provide more accurate forecasts of bond prices than existing dynamic term structure models, the accuracy of which is shown to be strongly dependent on the extraction method used to construct the term structure dataset used in the estimation.

The term structure extraction models in the literature estimate a large dimensional object through parametric techniques, such as McCulloch (1971, 1975), Vasicek and Fong (1982), Shea (1985), Steeley (1991), Mastronikola (1991), Waggoner (1997), Anderson and Sleath (2001), and non-parametric techniques, such as Fisher et al. (1995) and Linton et al. (2000). These techniques are designed to preserve the smooth shape of the term structure curve as well as its monotonic behaviour in a parsimonious way. The focus of these models is to fit directly either the discount curve in bond prices or the forward curve[3]. According to the literature[4], the most accurate way to fit the discount curve is through the usage of smooth splines. Bond prices then become linear in parameters and can be easily fitted by OLS. On the other hand, the cross sectional fit of the estimated curves, and its shape, becomes dependent on how many knots are used to approximate the discount function and on what type of spline functional form is used. Usually smoothing constraints are required to preserve the desirable monotonicity properties of the term structure and rule out negative forward rates. On the other hand, Nelson and Siegel (1987) proposed a model (NS) which assumes a specific functional form designed to fit the instantaneous forward rates curve which only depends on the term to maturity of each rate and four parameters, three of which will be

[3]The rate $F_t(\tau, \tau+\epsilon)$ is the τ forward ε rate at time t and in discrete time it is defined by the relation $[1+r_t(\tau+\epsilon)]^{(\tau+\epsilon)-t} = [F_t(\tau, \tau+\epsilon)+1]^{\epsilon}[1+r_t(\tau)]^{\tau-t}$, where $r_t(\tau+\epsilon)$ is the $\tau+\varepsilon$ spot rate at time t. It represents the marginal return in extending the investment in a zero-coupon bond with yield $r_t(\tau)$ maturing at τ for another ε months. For ε small is possible to obtain the instantaneous forward rate defined as $f_t(\tau) = -\frac{d_t'(\tau)}{d_t(\tau)}$. For a detailed treatment of forward rates and the previous relations see Hull (2014).

[4]See Steeley (2008) for a complete discussion on the various techniques to estimate the term structure.

interpreted as the level, slope and curvature of the term structure curve at each t[5]. Both approaches are popular in central banks and incorporated in their toolbox. Recently Koo et al. (2021), building up on Linton et al. (2000), introduced a non parametric estimator for the term structure which combines time series and cross-sectional information while extracting the term structure of interest rates from bond prices, which can ultimately also incorporate information from other macro variables.

Regarding the dynamic modelling of the term structure curves, the leading literature focuses on dynamic multi-factor models which provide a simpler statistical modelling framework for the entire curve. These models achieve dimensionality reduction as well as providing a clearer interpretation for the common movements of the interest rates at different maturities. This is consistent with several studies on the term structure, like Litterman and Scheinkman (1991), Willner (1996), Bliss (1996), Joslin et al. (2014), Jungbacker et al. (2014), which confirm the factor structure and that in most of the cases the first three principal components can describe more than 95% of the rates variation. Moreover this approach is consistent with the arbitrage-free theory of affine dynamic term structure model (DTSM)[6]. The current industry benchmark is the Dynamic Nelson-Siegel (DNS) model defined by Diebold and Li (2006) and it has been extended several times[7] including an arbitrage-free version by Christensen et al (2011). The DNS model fits the entire observed yield curve at once assuming that the parameters for level, slope and curvature in the Nelson and Siegel (1987) specification evolve dynamically according to a VAR(1) structure. Several extension of the model exists. A complete different approach is taken by Bowsher and Meeks (2008) who use a dynamic spline model, the Functional Signal plus Noise (FSN), described in Bowsher (2004) to model the term structure of interest rates. Its still a dynamic factor model with basis splines factor loadings dynamics driven by a cointegrating relation between the interest rates at the knots of the spline.

[5]The Nelson and Siegel (1987) model assumes that the instantaneous forward rate $f(\tau)$ is the solution of a second-order differential equation with real unequal roots, $f(\tau) = \beta_0 + \beta_1 e^{-\frac{\tau}{\lambda}} + \beta_2 \frac{\tau}{\lambda} e^{-\frac{\tau}{\lambda}}$ for a given value of λ. In this way forward rates are constrained to converge to a constant rate in the long run and reflect expectations about future short term interest rates. With this functional form it is possible to generate a family of curves that takes on monotonic or humped shapes depending on the values of the parameters. An extension of this model has been developed by Svensson (1994, 1995) including an additional parameter to give more flexibility.

[6]see Dai and Singleton (2000), Duffee (2002) among many others.

[7]For a comprehensive discussion on the DNS extensions see Härdle and Majer (2016).

The aim of all these approaches is to dynamically fit a functional form to an "observed" term structure curve which is statistically unrelated to the observed bond prices. As advocated by The estimation of these models needs an effective "two-step" procedure[8] to obtain term structure curves forecasts, which is unsatisfactory from an econometric standpoint compared to a single inferential framework. Moreover, neglecting the time dimension of the observed data in the extraction step, measurement errors in the cross-sectional fit of the extraction models could play a larger role in the forecasting step and affect the efficiency of the forecasts. Recent researchers such as Dai et al. (2004) and Fontaine and Garcia (2012) have argued that using forecasted synthetic interpolated yields can eliminate interesting dynamics by excessively smoothing bond price variation. This might ultimately introduce unnecessary measurement errors.

For these reasons Andreasen et al. (2019) proposed the idea of estimating ATSM directly on bond prices in a "single-step" inferential framework showing an improvement in the efficiency of the parameters estimation of the dynamic models. However, given that the relationship between bond prices and term structure is not linear, their "single-step" approach exploits the first order Taylor approximation of the Extended Kalman Filter. Moreover, it still assumes Gaussianity of the bond price errors, which can be a restrictive assumption. Given that extracting and modelling the term structure from bond prices can be considered as a signal extraction exercise, we will consider more sophisticated approaches for non linear signal extraction to tackle this problem.

The framework behind both the models introduced assumes that, given the entire cash-flows schedule, bond prices are directly driven by the unobserved discount function $d(\tau)$. Moreover, are observed with measurement error which captures both measurement and micro-structure noise. However, given the results of Gurkaynak et al. (2007), the measurement error is larger for longer maturity bonds, the cross-sectional heteroscedasticity also in assumed to be proportional to the inverse of the bonds duration. The first model extends the FSN dynamic spline model introduced by Bowsher (2004) and Bowsher and Meeks

[8] First to extract the term structure dataset from observed bond prices with one of the mentioned extraction models and then to use the data to estimate the parameters of the dynamic model.

(2008) to bond prices[9]. B-Splines are used to capture the unobserved discount function as in Steeley (1991) and their dynamics is driven by a dynamic factor structure on the spline coefficients. The advantage of a dynamic factor structure on the discount function is that, since it is linear in bond prices, it is possible to use signal extraction methods to estimate the discount function at every point in time through the means of the Kalman filter. This allow us also to obtain smoothed estimates of the discount function signal at every point in time and to make full use of all time series and cross-sectional information in the observed bond prices surface while producing forecasts. The cross-sectional heteroscedasticity at each time steps is approximately considered as proportional to the inverse of the bonds duration, as suggested by Gurkaynak et al. (2007). The second model extends the DNS model of Diebold and Li (2006) to bond prices assuming a Nelson-Siegel-Svensson (NSS) specification for the term structure as described by Svensson (1994, 1995). Then we use the score-driven approach introduced by Creal et al. (2013) and Harvey (2013) to give dynamics directly to the unobserved dynamic term structure factors filter out the dynamics of the unobserved rates. This non-linear filtering approach has been extensively used in various non linear setting For example, it may follow a Student's t distribution as in Harvey and Harvey and Luati (2014) and Linton and Wu (2020), an exponential generalized beta distribution, as in Caivano et al. (2016), a binomial distribution as in the vaccine example by Hansen and Schmidtblaicher (2021), or represented by a mixture, see Lucas and Schwaab (2019), for modelling realized volatility, see Harvey and Palumbo (2023), and circular time-series, see Harvey et al. (2023). For a comprehensive review of score-driven models see Harvey (2022). The optimality of score-driven models as a non linear filter for unobserved dynamic signals in observation-driven models is discussed in Blasques et al. (2015). Moreover, in doing so we can assume that the measurement error is both distributed as a multivariate Gaussian or multivariate t distribution. In addition, following the findings of Hautsch and Ou (2012) on bond yields consistent with the literature on stochastic volatility in term structure models, as in Harvey and Luati (2014) a separate dynamic factor is included to capture indepen-

[9]Dynamic splines were used by Harvey and Koopman (1993) to estimate the intra-weekly patterns of hourly electricity demand. In the same fashion they have been used by Ito (2013) to capture the dynamic diurnal patterns of intra-day high-frequency volume data. In the dynamic term structure literature splines were used in the context of dynamic factor models also by Jungbacker et al. (2014), which used splines to obtain smooth factor loadings for modelling the term structure.

dently heteroscedasticity across time. Moreover, as proposed by Koopman et al. (2010) and Quaedvlieg and Schotman (2020), we provide further flexibility in the NSS functional form over time giving dynamics also to the factor loadings. The introduced term structure models in the current format are not arbitrage-free but they are simply considered from an econometric standpoint as a signal extraction tool and in this fashion are compared with their competitors. The overall contributions of this paper so far can be then summarised as follows. The paper introduces two models which can provides forecasts of bond prices while forecasting and dynamically signal-extracting from observed bond prices the term structure of interest rates at every point in time in a single inferential framework using all the cross-sectional and time series information from observed bond prices. The models a coherent representation of the unobserved discount function and term structure of interest rates while achieving dimensional reduction as the cross-sectional dimension of the bond prices maturities becomes large. We provide extensions to the second model which allow more flexibility in capturing the stylised features of the term structure, in addition to heteroscedastocity in the time dimension and the zero-lowerbound. Moreover we also study the large samples properties of the estimated model parameters while establishing conditions for the stationarity and invertibility of the term structure filter.An assessment of their fitting and forecasting performance shows that the models' fit is comparable with other spline extraction models, while outperforming other existing dynamic term structure models in forecasting observed bond prices at one, six and twelve months horizons. Finally, the study highlights a strong sensitivity of the forecasting errors of the existing dynamic models to the choice of the term structure extraction method used to construct the samples for the parameters' estimation. The reminder of this paper is organised as follows. Section 2 presents the modelling assumptions of the two models as well as their estimation and model selection methods. Section 3 describes the dataset used and outlines the estimation results. Moreover assesses the out-of-sample performance of the models compared with other existing dynamic term structure models. Section 4 provides the conclusions.

2. Statistical Framework

In order to model bond prices directly this study assumes that the observed bond prices can be decomposed in an unobserved signal and a noise component.

Assuming a monthly t frequency, at any given point in time t a coupon bond j, with coupon frequency f_c and with a remaining maturity τ_j has $s+1$ cashflows, for $s = \lfloor (\tau_j - t) / f_c \rfloor$[10]. Its market price can be described as a combination of a noise process and the underlying unobserved price process which is itself a combination of the bond's discounted cashflow payments. To define the relation between the unobserved bond price process and the discount rates processes the following non-arbitrage relation for the bond price is assumed to hold[11]:

$$p_{jt} = \sum_{i=0}^{s} c_j \, d_t \left(\tau_{j-if_c}\right) + d_t \left(\tau_j\right) \tag{1}$$

where c_j are the coupon payments of the bond with monthly maturity τ_j and $d_t \left(\tau_{j-i\,f_c}\right)$ is the monthly discount function evaluated at the periodic maturities $\tau_{j-i\,f_c}$ with period f_c[12], for $i = 0, 1, \ldots, s$. From this relation it is clear that the bond price at t, p_{jt}, for $j = \{1, 2, , \ldots, N\}$, is a function only of the maturity τ_j, the discount function $\boldsymbol{\delta}_t = \left\{d_t \left(\tau_{j-sf_c}\right), \; d_t \left(\tau_{j-(s-1)f_c}\right), \ldots, d_t \left(\tau_j\right)\right\}$, and the coupon value c_j. For example, if at a given point in time t a large cross-section of semi-annual bonds, $f_c = 6$, is considered for a large number of N different consecutive maturities, such that with payment dates $\tau_1 < \tau_2 < \ldots < \tau_N$, the linear relation between the bond prices and the underlying discount function can be represented by the following system of linear equations,

$$\begin{array}{ccccccccc}
p_{1t} & = & c_1 d_t\left(\tau_1\right) & & & & & & +d_t\left(\tau_1\right) \\
p_{2t} & = & c_2 d_t\left(\tau_2\right) & & & & & & +d_t\left(\tau_2\right) \\
\vdots & = & \vdots & \ddots & & & & & \vdots \\
p_{7t} & = & c_7 d_t\left(\tau_1\right) + & c_7 d_t\left(\tau_7\right) & & & & & +d_t\left(\tau_7\right) \\
p_{8t} & = & c_8 d_t\left(\tau_2\right) + & c_8 d_t\left(\tau_8\right) & & & & & +d_t\left(\tau_8\right) \\
\vdots & = & \vdots & \vdots & \ddots & & & & \vdots \\
p_{N-1,t} & = & c_{N-1} d_t\left(\tau_6\right) + & c_{N-1} d_t\left(\tau_{13}\right) + & \ldots & +c_{N-1} d_t\left(\tau_{N-7}\right) & & & +d_{Nt}\left(\tau_{N-1}\right) \\
p_t & = & c_N d_t\left(\tau_1\right) + & c_N d_t\left(\tau_7\right) + & \ldots & +c_N d_t\left(\tau_{N-6}\right) & +c_N d_t\left(\tau_N\right) & & +d_t\left(\tau_N\right)
\end{array}$$

[10] Where $\lfloor x \rfloor = \{a \mid a \in \mathbb{Z} \wedge a \leq x\}$.

[11] For simplicity of notation we denote $p_{jt} = p_t\left(\tau_j\right)$ the price of the bond with maturity τ_j

[12] This depends on the coupon frequency of the bond, usually annual or semi-annual.

which reduced form becomes,

$$\mathbf{p}_t = (C_t + I)\,\boldsymbol{\delta}_t \tag{2}$$

where $\mathbf{p}_t = \{p_{1t}, p_{2t}, \ldots, p_{Nt}\}'$ is a vector of coupon bond prices at t with consecutive maturities up to N. Then, the semi-annual C_t matrix is a lower triangular matrix of the form,

$$C_t = \begin{pmatrix} c_{1,t} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_{2,t} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & c_{3,t} & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & c_{6,t} & 0 & 0 & 0 & 0 & \ldots & 0 \\ c_{7,t} & 0 & 0 & \ldots & 0 & c_{7,t} & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_{8,t} & 0 & \ldots & 0 & 0 & c_{8,t} & 0 & 0 & \ldots & 0 \\ 0 & 0 & c_{9,t} & \ldots & 0 & 0 & 0 & c_{9,t} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & c_{10,t} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N,t} & 0 & 0 & \ldots & 0 & c_{N,t} & 0 & 0 & 0 & \ldots & c_{N,t} \end{pmatrix}$$

constructed by the elements of vector $\mathbf{c}_t = \{c_{1t}, c_{2t}, \ldots, c_{Nt}\}'$ of the coupons of each bond j at time t. The vector $\boldsymbol{\delta}_t$ contains all the discount factors from the discount function at time t for each maturity τ_j, for j from 1 to N.

Assuming that Equation (2) is true, given a discount function $\boldsymbol{\delta}_t$ at a given point in time t the bond price vector will depend only on the structure of the coupon vector $\mathbf{c}_t$. The coupon vector $\mathbf{c}_t$ is deterministically time varying, and so the matrix C_t, since the bond prices dataset is sorted by months to maturity at every t. How this occurs can be described noticing that a bond with maturity τ_N observed at time t will have maturity $\tau_N - 1$ at time $t+1$. Assuming that the time step between each consecutive periods is equal to the time steps between the consecutive maturities τ_j, at time $t+1$ the bond will have maturity τ_{N-1}[13]. In order to accommodate for this effect in the dataset the vector $\mathbf{c}_t$ also varies over time in a deterministic fashion. As the bonds maturity shortens, at each time step the corresponding

[13]This effect is called by practitioners "roll-down" of the bond. Is the implicit effect of the time passing on the bond price which, as the maturity dates approaches, the bond's cashflows will be discounted by shorter, and therefore lower in a normal economic environment, rates and its price would get closer to 100.

coupon is moved down in the vector $\mathbf{c}_t$. Since the bond with maturity $\tau_1 = t$ at time t will mature at time $t+1$, the first coupon in the vector $\mathbf{c}_t$ is removed and the coupon of the new longest maturity bond, c_{Nt}, is added at the bottom of the vector.

2.0.1. Heteroscedastic Bond Prices Errors

Introducing a stochastic measurement error to Equation (2) we have,

$$\mathbf{p}_t = (C_t + I)\,\boldsymbol{\delta}_t + \boldsymbol{\varepsilon}_t \tag{3}$$

As highlighted by Bliss (1996) at every point in time we have a large variety of bonds with different cashflows and the longer the maturity of a bond the larger are its price fluctuations. This can be interpreted in heteroscedasticity in the bond prices cross-section at every t. To tackle this issue Gurkaynak et al. (2007) exploited the information in the duration of the bonds to capture the cashflows heterogeneity while extracting the term structure with the NSS model. Therefore, as it is done in Bliss (1996) to compare the MSE from fitting different extraction models to the bond prices cross-section, in fitting bond prices at every t they have used weighted least squares (WLS) weighting each squared residual by the inverse of the duration of the bond, $\left(p_t(\tau_i) - \hat{p}_t(\tau_i)\right)^2 / d_t(\tau_i) = \hat{\varepsilon}_{it}^2$. This makes sure to remove also the impact of other bond specific features, like the overall maturity length, the coupons size and trading periods on the fitted term structure curves.

This WLS approach controls for heteroscedasticity in the regression error assuming that the structure of the heteroscedasticity is proportional to the duration of the bonds, $\sigma_{it}^2 = d_t(\tau_i)$. This feature can be incorporated in our framework assuming that the variance-covariance matrix of the measurement error $\boldsymbol{\varepsilon}_t$ is time varying and proportional to the bonds durations. In order to do so we set $E\left[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'\right] = \Omega_{\varepsilon t} = \sigma_\varepsilon^2 D_t$, where σ_ε^2 is a time invariant parameter of the variance of the bond price errors, while D_t is a time varying diagonal matrix which has on the diagonal the elements of the vector $\mathbf{d}_t$ which is composed by the durations of the bonds in the cross-section for each time period t, sorted by maturity. As time passes, bond prices move along in the cross-section, as the coupons do in the $\mathbf{c}_t$ vector, and so the durations in the vector $\mathbf{d}_t$.

2.1. Bond-FSN Factor Model

The first model introduced is a version of the FSN defined by Bowsher (2004) applied to the the theoretical non-arbitrage relation between the term structure and the bond prices.

The present study assumes that the observed bond prices can be decomposed in a signal $\boldsymbol{\mu}_t$, described by the relation in Equation (2), and in a Gaussian microstructure noise $\boldsymbol{\varepsilon}_t$, as in Equation (3). The unobserved discount function signal $\boldsymbol{\delta}_t$ is modelled using a dynamic spline function uniquely determined by the time varying vector of spline coefficients $\boldsymbol{\gamma}_t$[14]. The time dynamics of $\boldsymbol{\gamma}_t$ is then modelled with a Dynamic Factor Model (DFM) driven by the vector of factors $\boldsymbol{\mu}_t$ of relatively small dimension compared to the entire signal discount function.

Given an N-dimensional vector of bond maturities $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, , \tau_{N-1}, \tau_N)'$ and an observation vector of coupon bond prices $\mathbf{p}_t$, the bond FSN model can be represented as follows,

$$\mathbf{p}_t = \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim N\left(\mathbf{0}, \Omega_{\varepsilon t}\right) \tag{4}$$

$$\boldsymbol{\mu}_t = \left(C_t + I\right) \boldsymbol{\delta}_t \tag{5}$$

$$\boldsymbol{\delta}_t = S_{\gamma,t}\left(\boldsymbol{\tau}\right) \tag{6}$$

$$\boldsymbol{\gamma}_t = \bar{\boldsymbol{\gamma}} + \boldsymbol{\Pi}\boldsymbol{\alpha}_t + \boldsymbol{\xi}_t, \qquad \boldsymbol{\xi}_t \sim N\left(\mathbf{0}, \Omega_{\xi}\right) \tag{7}$$

$$\boldsymbol{\alpha}_t = \boldsymbol{\Psi}\boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim N\left(\mathbf{0}, \Omega_{\eta}\right) \tag{8}$$

where $\boldsymbol{\mu}_t$ is the N-dimensional unobserved location signal. $S_{\gamma,t}\left(\tau\right)$ is the N-dimensional spline based on m parameters. $\boldsymbol{\gamma}_t$ is the m-dimensional vector of spline parameters, $\bar{\boldsymbol{\gamma}} \in \mathbb{R}^m$ and $\boldsymbol{\alpha}_t$ is the q-dimensional vector of dynamic factors. For the model to be well specified it is necessary that $q < m < N$. $\boldsymbol{\varepsilon}_t$, $\boldsymbol{\xi}_t$ and $\boldsymbol{\eta}_t$ are multivariate Gaussian disturbances distributed with mean $E\left[\boldsymbol{\varepsilon}_t\right] = E\left[\boldsymbol{\xi}_t\right] = E\left[\boldsymbol{\eta}_t\right] = 0$. The covariance matrices are $\Omega_{\varepsilon t} = \sigma_{\varsigma}^2 D_t$, where $D_t \in \mathbb{R}^{N\times N}$ is the heteroschedastic duration matrix, while $\Omega_{\xi} \in \mathbb{R}^{m\times m}$ and $\Omega_{\eta} \in \mathbb{R}^{q\times q}$ are time

[14] In order simplify the identification of the $\boldsymbol{\delta}_t$ vector in Equation (2) uniquely, restrictions on the C_t matrix are imposed. The coupons in the vector $\mathbf{c}_t$ of the missing bonds in the cross-section at every given t are replaced by the average of the coupons of all the bonds available at t. Since we are using the Kalman Filter and the prediction error decomposition to estimate the parameters of the model and the unobserved signal, this restriction is not strictly necessary since the Kalman Filter can handle observation vectors with different dimensions, however it is computationally handy and contributes to maintain a decent speed for the optimization algorithm.

invariant. The disturbances are uncorrelated at every t, $E\left[\boldsymbol{\eta}_t \boldsymbol{\varepsilon}_t'\right] = E\left[\boldsymbol{\eta}_t \boldsymbol{\xi}_t'\right] = E\left[\boldsymbol{\xi}_t \boldsymbol{\varepsilon}_t'\right] = 0$. The matrices $\mathbf{\Pi} \in \mathbb{R}^{m \times q}$ and $\mathbf{\Psi} \in \mathbb{R}^{q \times q}$ are respectively the dynamic parameters matrix and the factor loading matrix. For the model to be identified for estimation also other restrictions on these system matrices need to be imposed[15].

Assuming that the term structure and the discount function share the same time properties, a factor dynamics for the discount function is consistent with the dynamic multi-factor literature on term structure modelling.

The model is linear in the time varying parameters, so can be put in state space form and estimated through the mean of the Kalman Filter[16]. The likelihood can be constructed through the prediction error decomposition and parameters can be estimated through numerical maximum likelihood optimizations. This paper uses the BFGS algorithm to perform the optimization[17].

2.1.1. Spline Functions

The spline function S is defined as the standard linear combination of smooth functions $S_j(x)$ over an interval $[a,\, b,]$ partitioned in m sub-intervals, delimited by $m+1$ consecutive points $\mathbf{k} = \{k_1 = a,\, k_2,\, \ldots,\, k_m,\, k_{m+1} = b \mid k_1 < \ldots < k_{m+1}\}$, the spline knots,

$$S(x) = \sum_{j=1}^{m} \lambda_j S_j(x)\, \mathbb{I}_{[k_j,\, k_{j+1}]}(x)$$

where $\boldsymbol{\gamma} = \{\gamma_1,\, \gamma_2,\, \ldots,\, \gamma_{m-1},\, \gamma_m\}$ are the coefficients. In our case is used to approximate the discount function $\boldsymbol{\delta}$ in bond prices at a given set of maturities $\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots, \tau_N\}$.

$$\boldsymbol{\delta} = \mathbf{W}\boldsymbol{\lambda}$$

[15] For a detailed discussion of identification in Dynamic Factors Models see Harvey (1993).

[16] The presence of a time varying C_t doesn't invalidate the Kalman Filter since it varies only deterministically, also in the future. On the other hand $\Omega_{\varepsilon t} = \sigma_{\varsigma}^2 D_t$ depends on the duration matrix D_t which only partially varies deterministically. This would imply that $\mathbf{p}_t$ is only conditionally Gaussian. However, in this case the Kalman Filter derivations still remain valid and the model can still produce unbiased one-step-ahead predictions, see Harvey (1993). In this way the prediction errors $\mathbf{v}_t = \mathbf{p}_t - \boldsymbol{\mu}_t$ in the Kalman recursion are weighted by the cross-sectional duration of each bond achieving a similar result as the one of the WLS of Gurkaynak et al. (2007).

[17] See Nocedal and Wright (1999). Computationally efficient versions of the Kalman filter that can be used in this context have been developed for multivariate models by Koopman and Durbin (2000). Further improvements in the computational time can be achieved for dynamic factor models using the methods of Jungbacker and Koopman (2015).

here $\boldsymbol{\delta} = \{\delta(\tau_1), \delta(\tau_2), \ldots, \delta(\tau_N)\}$ is the sequence of discount function ordinates and $\mathbf{W}$ is the $N \times m$ matrix of spline functions evaluated at each point in $\boldsymbol{\tau}$. In the dynamic splines literature[18] a dynamics is specified for the spline coefficients $\boldsymbol{\gamma}_t$, making the spline function a dynamic factor model with fixed factor loadings W deterministic over time.

$$\boldsymbol{\delta}_t = \mathbf{W}\boldsymbol{\gamma}_t$$

The m dynamic spline parameters $\boldsymbol{\gamma}_t$ are then modelled with another layer of unobserved dynamic factor. As a result, this nested dynamic factor model has an analogous interpretation to the local linear trend model in Harvey (1993)[19].

Through the usage of spline functions bond prices are linear in parameters and factors showing clearly the relation between the dynamics of rates and bond. Moreover, since the unobserved dynamics can linearly model the observed bond price surface, it allows the use of exact filtering techniques for signal extraction and parameter estimation.

For the construction of the $\mathbf{W}$ matrix in the FSN model, instead of using the cubic splines of Poirier (1976) and Monahan (2001) as in Bowsher (2004) and Bowsher and Meeks (2008), the present work uses the Basis Splines (B-Splines) used by Shea (1984) and Steeley (1991, 2008) which, are shown empirically to approximate better than other splines the discount function of interest rates. This is because, as shown by Steeley (2008), compared to other spline functions specifications, the higher stability in the construction of B-splines better preserves the discount function features without imposing any smoothness or monotonicity constraints other than $d(0) = \sum_{j=1}^{m} \lambda_j B_j(0) = 1$. This restriction imposes the spline to have

[18] For example Harvey et al. (1997), Bowsher and Meeks (2008) and Jungbacker et al. (2014).

[19] In order to give an interpretation to the dynamics of the spline coefficients, since the $\mathbf{W}$ matrix is fixed, it is possible to obtain a factor rotation in order to restrict m rows of $\mathbf{W}$ at the spline knots to be equal to the subsequent rows of the $m \times m$ identity matrix $\mathbf{I}_m$, as described in the appendix. In this way the parameters $\boldsymbol{\gamma}_t$ driving the dynamics of the spline can be exactly matched with a subset of discount rates among the spline cross-section. This factor rotation is purely for identifying the unobserved dynamics of the spline coefficients and it is arbitrarily chosen in order to match the unobserved spline coefficients with discount rates. The rotation ultimately affects neither the estimation nor the forecasting results

value 1 at maturity 0 and can be easily imposed in our model[20]. Moreover the B-splines specification allows us to specify a smaller number of knots in $\mathbf{k}$ than the number of dynamic parameters in $\boldsymbol{\gamma}_t$, which simplifies the knots selection procedure. At last, B-splines allow for a more flexible spline function through the possibility of setting the smoothness degree k of the approximating curves as an additional degree of freedom, which gives the possibility to explore fitting the discount function also with higher degree splines than $k = 3$[22].

2.1.2. Knot Selection

In the spline literature there are various approaches for selecting the number and location of the knots. In the non parametric literature there are several model specification tests for series methods which compare the cross-sectional fit penalizing for the parameters dimension in a similar way to the Schwartz information criteria (SIC), like He et al. (2001) and Mao and Zhao (2003). In the term structure estimation literature knot selection has mostly been performed either using rules of thumb or looking heuristically at the overall shape of the estimated discount function[23]. In the context of dynamic term structure models, Bowsher and Meeks (2008) compared the in-sample fit[24] of the model across all possible set of knots. On the other hand Jungbacker et al. (2014) have developed a series of rigorous tests to infer from the overall estimated likelihood of their model the optimal number and locations of spline knots. These tests are fairly general and can be used in the present framework.

Drawing on both the non parametric and the term structure estimation literature, the present paper suggests the following iterative procedure to define the $l = n + 1$ number of knots and the vector $\mathbf{k}$ of knots positions. The procedure is based on both cross-sectional in-sample fit and the estimated likelihood of the model. Both these criteria are used because,

[20] This restriction can be imposed while using the Kalman filter following the approach of Doran (1992). Each observation vector $\mathbf{p}_t$ is augmented of an additional observation in the cross-section as the first line of the transition equation, for $\tau = 0$[21]. Then the transition equation for this series becomes $\sum_{j=1}^{m} \gamma_{jt} S_j\left(0\right) = 1$ at every t. While assuming that this additional series has a Gaussian uncorrelated measurement error such that $E\left[\varepsilon_t\left(0\right)\right] = 0$ and $E\left[\varepsilon_t\left(0\right)^2\right] = 0$ the constraint is guaranteed to hold in both the updating and smoothing equations of the Kalman filter.

[22] However, as noted by Steeley (2008), higher degrees splines can better fit the bond price data but the resulting term structure can become less smooth.

[23] For example McCulloch (1971) suggested to set the number of knots to the nearest integer to $\sqrt{N}$ with N equal to the number of bonds available at a given point in time. The knots positions should be equally spaced across the maturity domain. On the other hand Litzenberger and Rolfo (1984) suggested only to use up to 5 knots and to place three of them at the maturities corresponding to 1y, 5y and 10y.

[24] In terms of minimising the residual sum of squares (RSS).

as noted by Bowsher and Meeks (2008), the ability to mimic the shape of the underlying discount function signal is a necessary preliminary property for the FSN model to perform well in forecasting. The procedure starts assuming an initial number of knots l_s.

In the first stage for each number of knots l_s a large number of static splines with different knot vectors $\mathbf{k}_s \in \mathbb{R}^{l_s}$ are constructed with all the possible ordered combinations of l_s maturities in $\boldsymbol{\tau}$. These are fitted through OLS to the discount function of each of the T observed price vectors, $\mathbf{p}_t(\boldsymbol{\tau})$. Then, the average across time of the residual sum of squares (RSS) from each of the cross-sectional regressions is compared, and the vector $\mathbf{k}_s$ which gives the smallest RSS is selected[25].

In order to preserve the spline consistency of the FSN model it is essential to focus on knot vectors which always include the shortest and longest maturities of the discount function, $\tau = 1$ and $\tau = 84$. Then all the remaining knots need to be lying in the set $\bar{\boldsymbol{\tau}} = \{3, ..., 11, 12, 13, 14, ..., 79, 80, 82\}$ and cannot be consecutive to one another[26]. While using B-Splines with degree k the smallest number of knots to consider is $l_0 = 2k + 1$.

Once the optimal location vector $\mathbf{k}_s^*$ of l_s knots is determined, the same procedure is run again for a number of knots equal to $l_{s+1} = l_s + 1$ in order to obtain also the optimal vector $\mathbf{k}_{s+1}^*$ for l_{s+1} knots. Then, in the second stage, the log-likelihoods of the two models for $\mathbf{k}_s^*$ and for $\mathbf{k}_{s+1}^*$ are then compared using a likelihood ratio test which takes the inspiration from Jungbacker et al. (2014)

$$LR_s = -2\left(\hat{L}\left(\mathbf{k}_s^*\right) - \hat{L}\left(\mathbf{k}_{s+1}^*\right)\right) \sim \chi_1^2$$

Here $\hat{L}\left(\mathbf{k}_s^*\right)$ is the log-likelihood at the restricted optimal set of knots $\mathbf{k}_s^*$ while $\hat{L}\left(\mathbf{k}_{s+1}^*\right)$ is the log-likelihood at the unrestricted optimal set of knots $\mathbf{k}_{s+1}^*$. As long as $LR_s > \chi_1^{2-1}(0.95)$ the tests rejects the restricted knots vector $\mathbf{k}_s^*$ in favour of the unrestricted $\mathbf{k}_{s+1}^*$. Then the procedure restarts from the beginning with $l_s = l_{s+1} + 1$. When the restricted set of vectors doesn't get rejected the model procedure stops and the optimal knots vector for the model becomes $\mathbf{k}^* = \mathbf{k}_s^*$.

In this way the inclusion of the information in the estimated log-likelihoods allows us

[25] This first stage knots selection procedure is analogous to the one used by Bowsher and Meeks (2008).

[26] As showed by Steeley (1991); McCulloch (1971), the usage of consecutive knots in estimating the term structure can lead to numerical instability in the fitted splines.

to perform a model selection taking into account the increasing dimensionality given by the inclusion of additional knots. With this approach it becomes computationally feasible to search for the optimal knots vector over a large model space.

2.2. Bond-DNSS Score-Driven Model

The Bond-FSN model tackles the estimation and modelling of the term structure extracting the discount approximated function signal with dynamic splines. The model then becomes linear in the time varying parameters and a Gaussian filter based on the Kalman Filter can be used to extract the unobserved signal. However, the accuracy of this approach relies on the accuracy of the spline function at approximating the unobserved discount function. Despite being quite flexible at capturing the smoothness of the function, and avoiding negative forward rates, still in this form it cannot tackle the negativity of the rates, and therefore issues like the zero-lowerbound, without complex non linear restrictions. Moreover, despite being robust to the term premia, and coupon size, impact on bond prices still is not robust to wider fluctuation and outliers as well as not being able to capture the heteroscedastocity in the time dimension.

The second proposed model takes into account directly the non linearity in the relationship between bond prices and the term structure. Moreover, we relax the assumption of Gaussianity assuming that the measurement error $\boldsymbol{\varepsilon}_t$ is conditionally distributed with a multivariate *location-and-scale* distribution G with location $\mathbf{0}$, scale matrix $\Omega_{\varepsilon t}$ and additional shape parameters $\boldsymbol{\psi}$. As a consequence also $\mathbf{p}_t$ is conditionally distributed with a multivariate distribution G. The model can be represented as follows,

$$\mathbf{p}_t = \boldsymbol{\mu}_{t|t-1} + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t|\mathcal{F}_{t-1} \sim G\left(\mathbf{0}, \Omega_{\varepsilon t}; \boldsymbol{\psi}\right) \tag{9}$$

$$\boldsymbol{\mu}_{t|t-1} = (C_t + I) \cdot \boldsymbol{\delta}_{t|t-1} \tag{10}$$

$$\boldsymbol{\delta}_{t|t-1} = \exp\left(-\mathcal{T} \cdot \mathbf{r}_{t|t-1}\right) \tag{11}$$

$$\mathbf{r}_{t|t-1} = \Lambda \cdot \boldsymbol{\beta}_{t|t-1} \tag{12}$$

The G distribution has as multivariate location $\boldsymbol{\mu}_{t|t-1}$ which depends both on the deterministically time varying coupon matrix C_t and the time varying discount factors $\boldsymbol{\delta}_{t|t-1}$. The discount factors assume exponential discounting and in turn depends non linearly on the interest rates vector $\mathbf{r}_{t|t-1}$. Then $\mathcal{T} \in \mathbb{R}^{N\times N}$ is a diagonal diagonal matrix with the maturities vector $\boldsymbol{\tau}$ on the main diagonal. The G distribution also has as additional parameters a scale matrix $\Omega_{\varepsilon t} = \sigma_{\varepsilon}^2 D_t$, where the cross-sectional heteroscedastocity is governed by the inverse of the duration of the bond prices matrix D_t as before, and a static vector $\boldsymbol{\psi}$ of distribution specific shape parameters. Following Svensson (1994, 1995), the interest rates vector $\mathbf{r}_{t|t-1}$ in the location $\boldsymbol{\mu}_{t|t-1}$ are modelled with a 5 factors DNSS specification. Then $\boldsymbol{\beta}_{t|t-1} \in \mathbb{R}^5$ is the vector of time-varying factors while $\Lambda \in \mathbb{R}^{N\times 5}$ is the DNSS factor loading matrix dependent on the two parameters λ_1 and λ_2 which model the curvature points of the term structure for the short and the long term maturities. The matrix is constructed as $\Lambda = (\boldsymbol{\lambda}_1', \boldsymbol{\lambda}_2', \dots, \boldsymbol{\lambda}_N')'$, where each of the j row vectors is defined as

$$\boldsymbol{\lambda}_j\left(\lambda_1, \lambda_2\right) = \left(\begin{array}{ccccc} 1, & \frac{1-e^{-\lambda_1 \tau_j}}{\lambda_1 \tau_j}, & \frac{1-(1+\lambda_1 \tau_j)e^{-\lambda_1 \tau_j}}{\lambda_1 \tau_j}, & \frac{1-e^{-\lambda_2 \tau_j}}{\lambda_2 \tau_j}, & \frac{1-(1+\lambda_2 \tau_j)e^{-\lambda_2 \tau_j}}{\lambda_2 \tau_j} \end{array}\right), \quad j = 1, ..., N$$

As previously done by Koopman et al. (2017) for modelling term structure time series, we can introduce time-variation in the $\boldsymbol{\beta}_{t|t-1}$ using a score-driven approach. Then the dynamics can be specified as

$$\boldsymbol{\beta}_{t+1|t} = \left(I - \Phi_\beta\right)\boldsymbol{\omega}_\beta + \Phi_\beta \boldsymbol{\beta}_{t|t+1} + K_\beta \mathbf{u}_{\beta t},$$

where $\boldsymbol{\omega}_\beta \in \mathbb{R}^{5\times 1}$. $\Phi_\beta \in \mathbb{R}^{5\times 5}$ and $\mathbf{K}_\beta \in \mathbb{R}^{5\times 5}$ are parameter matrices to be estimated. Finally the dynamics are driven by the conditional scores with respect to the time varying factors.

$$\mathbf{u}_{\beta t} = S_{\beta t} \frac{\partial \boldsymbol{\mu}'}{\partial \boldsymbol{\beta}} \frac{\partial \ln f_t}{\partial \boldsymbol{\mu}'}$$

Where

$$\frac{\partial \boldsymbol{\mu}'}{\partial \boldsymbol{\beta}} = -\Lambda' \mathcal{T} \operatorname{diagrv}\left(\exp\left(-\mathcal{T}\Lambda \boldsymbol{\beta}_{t|t-1}\right)\right)\left(C_t + I\right)' \tag{13}$$

$S_{\beta t}$ is a scaling matrix that often in score-driven models is set as $S_{\beta t} = \mathbf{I}_{\beta\beta}^{-1}$ where $\mathbf{I}_{\beta\beta} = \frac{\partial \boldsymbol{\mu}'}{\partial \boldsymbol{\beta}} \mathbf{I}_{\mu\mu} \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\beta}'}$ where $\mathbf{I}_{\mu\mu}$ is the information matrix with respect to the time varying location of the multivariate distribution.

The specification of the model is complete once we specify $\frac{\partial \ln f_t}{\partial \boldsymbol{\mu}'}$ and $\mathbf{I}_{\mu\mu}$. These depends on the choice of the distribution G. In the standard case considered so far, if the distribution G multivariate normal $\boldsymbol{\psi}$ is an empty vector and $\ln f_t$ is the log multivariate Gaussian density of $\mathbf{p}_t$. Then

$$\frac{\partial \ln f_t}{\partial \boldsymbol{\mu}'} = \Omega_{\varepsilon t}^{-1} \boldsymbol{\varepsilon}_t, \qquad \mathbf{I}_{\mu\mu} = -E\left[\frac{\partial^2 \ln f_t}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'}\right] = \Omega_{\varepsilon t}^{-1} \tag{14}$$

where $\boldsymbol{\varepsilon}_t = \mathbf{p}_t - \boldsymbol{\mu}_{t|t-1}$. This assumption is in line with the existing term structure literature which assumes Gaussianity. However, given that bond prices tends to have a much more non linear behaviour than smooth term structure series, in the score-driven framework we can assume that $\mathbf{p}_t$ can be instead distributed with a multivariate students t distribution. Then $\boldsymbol{\psi} = \upsilon$ the degrees of freedom of the students t and,

$$\frac{\partial \ln f_t}{\partial \boldsymbol{\mu}'} = \frac{\upsilon + N}{\upsilon}\left(1 - b_t^{\dagger}\right)\Omega_{\varepsilon t}^{-1}\boldsymbol{\varepsilon}_t, \qquad \mathbf{I}_{\mu\mu} = -E\left[\frac{\partial^2 \ln f_t}{\partial \boldsymbol{\mu}\partial \boldsymbol{\mu}'}\right] = \frac{\upsilon + 2}{\upsilon + N + 2}\Omega_{\varepsilon t}^{-1} \tag{15}$$

where $b_t^{\dagger} = \frac{\left(\boldsymbol{\varepsilon}_t'\Omega_{\varepsilon t}^{-1}\boldsymbol{\varepsilon}_t\right)/\upsilon}{1+\left(\boldsymbol{\varepsilon}_t'\Omega_{\varepsilon t}^{-1}\boldsymbol{\varepsilon}_t\right)/\upsilon}$ is distributed with a $Beta\left(\frac{N}{2}, \frac{\upsilon}{2}\right)$. The multivariate Gaussian score then is a special case of the multivariate t when $\upsilon \to \infty$. However, the advantage of assuming a multivariate t distribution is that then the score $\mathbf{u}_{\beta t}$ becomes robust to the presence of outliers as discussed by Harvey and Luati (2014) and D'Innocenzo et al. (2021). This comes handy if we see the term structure estimation as a signal extraction problem and we are looking for a more efficient filter able to capture the dynamics of the unobserved term structure factors. Moreover, as showed by Blasques et al. (2015), the score-driven has the property of optimal filter among the class of non linear observation-driven models. For its ability of dealing with non linearities, the use of a score-driven model is clearly more appropriate in extracting and modelling the term structure directly from bond prices rather than modelling smooth extracted term structure series as in Koopman et al. (2017). As described in Creal et al. (2013) and Harvey (2013), the parameters estimation of the model can be easily performed by numerical optimization algorithms through maximum likelihood.

2.2.1. heteroscedastocity Across Time

The choice of making the scale matrix $\Omega_{\varepsilon t}$ proportional to the time varying matrix D_t of bond durations is driven by the evidences of Gurkaynak et al. (2007) and it partially describes the cross-sectional heteroscedastocity of bond prices series derived by different term premia and coupon sizes. However, despite capturing some time variation, doesn't depend on any time heteroschedastic features of the observed time series. Many studies, such as Bianchi et al. (2009), Koopman et al. (2010), and Hautsch and Ou (2012), Koopman et al. (2017), Quaedvlieg and Schotman (2020) allow for a stochastic time variation in the variances of the DNS model while modelling term structure dataseries, while even more studies in the affine term structure literature include stochastic volatility.In our framework with can give a dynamics directly to the scaling factor σ_ε^2. Like in an EGARCH we can specify the parameter $h = \log \sigma_\varepsilon^2$ and specify a score-driven dynamics for $h_{t|t-1}$. This approach would be equivalent to the single volatility factor of the DNS model proposed by Koopman et al. (2017) for modelling the term structure dataseries. They also suggested alternatives with different volatility factors depending for various range of maturities, for example short, mid, and long. This can be easily implemented in our framework. However, given that in this case the observations are bond prices series, the presence already of the deterministic time varying duration matrix D_t can make the choice of the number of factor quite arbitrary. To maintain parsimony we prefer to stick to the single factor specification. As a result $\Omega_{\varepsilon t|t-1} = e^{h_{t|t-1}} \cdot D_t$, where $h_{t|t-1}$ would represents the time varying heteroscedastic component of the scale matrix, or in some sense the overall level of the volatility, while D_t is the purely the cross-sectional. Then the score-driven dynamics of $h_{t|t-1}$ can be specified as ,

$$h_{t+1|t} = (1 - \phi_h)\,\omega_h + \phi_h h_{t|t+1} + \kappa_h u_{ht},$$

where ω_h, ϕ_h and κ_h are additional parameters to be estimated and

$$u_{ht} = S_{ht}^{-1} \frac{\partial \ln f_t}{\partial h} \exp\left(h_{t|t-1}\right),$$

the scaling matrix can be set as $S_{ht} = \mathbf{I}_{hh}^{-1}$. Now if $\mathbf{p}_t$ is distributed with a multivariate Gaussian then,

$$\frac{\partial \ln f_t}{\partial h} = \frac{\boldsymbol{\varepsilon}_t' \Omega_{\varepsilon t|t-1}^{-1} \boldsymbol{\varepsilon}_t - N}{2}, \qquad \mathbf{I}_{hh} = -E\left[\frac{\partial^2 \ln f_t}{\partial h^2}\right] = \frac{N}{2}.$$

, where $\boldsymbol{\varepsilon}_t = \mathbf{p}_t - \boldsymbol{\mu}_{t|t-1}$, while if on the other hand we assume that $\mathbf{p}_t$ is distributed with a multivariate t then,

$$\frac{\partial \ln f_t}{\partial h} = \frac{\upsilon + N}{\upsilon} b_t^\dagger - \frac{N}{2}, \qquad \mathbf{I}_{hh} = -E\left[\frac{\partial^2 \ln f_t}{\partial h^2}\right] = \frac{N\upsilon}{2\upsilon + 2N + 4}.$$

where $b_t^\dagger$ is *Beta* distributed as before.

The current scale specification assumes $\Omega_{\varepsilon t|t-1}$ diagonal, therefore no contemporaneous correlation between bond prices errors. On the other hand if we want to relax this assumption a specification for a full matrix scale matrix can be also implemented as $\Omega_{\varepsilon t} = \sigma_t^2 D_t^{\frac{1}{2}} R D_t^{\frac{1}{2}}$. Then is possible to provide a score-driven dynamics to the correlation matrix R following the approach of Creal and Lucas (2011) as described in Koopman et al. (2017) in modelling the term structure dataseries. Given that the term structure extraction literature assumes uncorrelated bond prices, we have maintain this assumption in our study and we leave this extension to further research.

2.2.2. Time Varying Factor Loadings

As Nelson and Siegel (1987) and Svensson (1994, 1995) highlighted, the λ_i parameters governs the flexibility of the NS and NSS specifications identifying the maturity curvature points of the term structure. Often these parameters have been calibrated while using the DNS, see Diebold and Li (2006). However if estimated they tend to vary significantly across time periods. Koopman et al. (2010) attempted to model the time variation in λ_i using a Kalman filter. This implied time varying factor loadings in the $\boldsymbol{\beta}_{t|t-1}$ dynamics. Quaedvlieg and Schotman (2020) found that the NSS factor loadings have dramatically changed after the start of the financial crisis and provided a score-driven version of the DNSS with time varying λ_i to better capture long term interest rates movements.

In the spirit of suggesting the most flexible extraction and modelling framework we also included this feature in our model. In our case a score driven model for the λ_i parameters can be constructed as

$$\lambda_{lt+1|t} = (1-\phi_{l\lambda})\,\omega_{l\lambda} + \phi_{l\lambda}\lambda_{lt|t+1} + \kappa_{l\lambda} u_{lt}^{\lambda}, \qquad l = 1,2$$

where $\omega_{l\lambda}, \omega_{2\lambda}, \phi_{l\lambda}, \phi_{2\lambda}, \kappa_{1\lambda}$ and $\kappa_{2\lambda}$ are additional parameters to be estimated and

$$u_{lt} = S_{lt}^{-1} \frac{\partial \boldsymbol{\mu}'}{\partial \lambda_l} \frac{\partial \ln f_t}{\partial \boldsymbol{\mu}'}, \qquad l = 1,2$$

where,

$$\frac{\partial \boldsymbol{\mu}'}{\lambda_l} = -\boldsymbol{\beta}'_{t|t-1} \frac{\partial \Lambda'_{t|t-1}}{\partial \lambda_l} \mathcal{T} \operatorname{diagrv}\left(\exp\left(-\mathcal{T}\Lambda_{t|t-1}\boldsymbol{\beta}_{t|t-1}\right)\right)(C_t + I)', \qquad l = 1,2$$

while each row j of the matrix $\frac{\partial \Lambda_{t|t-1}}{\partial \lambda_1}$ is constructed as

$$\frac{\partial \boldsymbol{\lambda}_{jt|t-1}}{\partial \lambda_1} = \left(\; 0, \quad \frac{\left(1+\lambda_{1t|t-1}\tau_j\right)e^{-\lambda_{1t|t-1}\tau_j}-1}{\lambda^2_{1t|t-1}\tau_j}, \quad \frac{\left(1+2\lambda_{1t|t-1}\tau_j\right)e^{-\lambda_{1t|t-1}\tau_j}-1}{\lambda^2_{1t|t-1}\tau_j}, \quad 0, \quad 0 \; \right), \qquad j = 1, ..., N$$

and each row j of the matrix $\frac{\partial \boldsymbol{\Lambda}_{t|t-1}}{\partial \lambda_2}$ is constructed as

$$\frac{\partial \boldsymbol{\lambda}_{jt|t-1}}{\partial \lambda_2} = \left(\; 0, \quad 0, \quad 0, \quad \frac{\left(1+\lambda_{2t|t-1}\tau_j\right)e^{-\lambda_{2t|t-1}\tau_j}-1}{\lambda^2_{2t|t-1}\tau_j}, \quad \frac{\left(1+2\lambda_{2t|t-1}\tau_j\right)e^{-\lambda_{2t|t-1}\tau_j}-1}{\lambda^2_{2t|t-1}\tau_j} \; \right), \qquad j = 1, ..., N$$

Also in this case the scaling matrices $S_{lt} = \mathbf{I}^{-1}_{l\lambda\lambda}$, where $\mathbf{I}_{l\lambda\lambda} = \frac{\partial \boldsymbol{\mu}'}{\partial \lambda_l}\mathbf{I}_{\mu\mu}\frac{\partial \boldsymbol{\mu}}{\partial \lambda_l}$ for $l = 1, 2$, while $\partial \ln f_t / \partial \boldsymbol{\mu}'$ and $\mathbf{I}_{\mu\mu}$ depend on the multivariate distribution G of the price multivariate bond prices vector $\mathbf{p}_t$ as described in Equation (14) and Equation (15).

2.2.3. Zero-Lowerbound

More recently the term structure literature has been focused on capturing the behaviours of interest rates as they approach 0. To do so the literature has shifted towards incorporating the idea of the shadow rate as proposed by Black (1995). He assumed that assumed that the short rate r_t that governs the term structure cannot fall below a lowerbound r_{LB} due to the presence of a physical currency with a natural interest rate of zero. Therefore there is a shadow short rate s_t which would prevail in a world without the option of physical currency such that $r_t = \max(r_{LB}, s_t)$. Notable papers which exploited this approach are Krippner (2015), Wu and Xia (2016), Christensen and Rudebusch (2014) among others. On the other hand Opschoor and Van der Wel (2022) have generalised this idea directly on to the yield curve such that,

$$r_{LBit} = r_{LB} + \max\{0, r_{it} - r_{LB}\}$$

This can then be applied to all the rates in the term structure in the following way,

$$\mathbf{r}_{LBt} = r_{LB}\mathbf{1} + \max\{\mathbf{0}, \mathbf{r}_t - r_{LB}\mathbf{1}\}$$

Morever, Opschoor and Van der Wel (2022) notice that the max function can be approximated as $\max\{0, x\} \approx \varphi g\left(\frac{x}{\gamma}\right)$ where φ is *flexbility* parameter. The re are many functions g

which would approximate the max function, however they suggest the following,

$$
g(x) = \begin{cases} \ln(1+e^x) \\ x\Phi(x) + \phi(x) \end{cases} \tag{16}
$$

where $\Phi(x)$ and $\phi(x)$ are respectively the cdf and pdf of the standard normal distribution. If we want to incorporate this feature in our model we can rewrite Equation (12) in the following way

$$
\mathbf{r}_{LBt|t-1} = r_{LB}\mathbf{1} + \varphi g\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right]
$$

Then the score functions in Equation (13) and Section 2.2.2 will be modified as follows,

$$
\begin{aligned}
\frac{\partial\boldsymbol{\mu}'}{\partial\boldsymbol{\beta}} =& -\Lambda'_{t|t-1}\mathrm{diagrv}\left\{g'\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right]\right\}\mathcal{T}\cdot \\
&\cdot\mathrm{diagrv}\left(\exp\left\{-r_{LB}\mathbf{1} - \varphi\mathcal{T}g\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right]\right\}\right)(C_t+I)' \\
\frac{\partial\boldsymbol{\mu}'}{\lambda_l} =& -\boldsymbol{\beta}'_{t|t-1}\frac{\partial\Lambda'_{t|t-1}}{\partial\lambda_l}\mathrm{diagrv}\left\{g'\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right]\right\}\mathcal{T}\cdot \\
&\cdot\mathrm{diagrv}\left(\exp\left\{-r_{LB}\mathbf{1} - \varphi\mathcal{T}g\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right]\right\}\right)(C_t+I)', \qquad l=1,2
\end{aligned}
$$

where if we use the first approximation in Equation (16) we have that

$$
\begin{aligned}
g\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right] =& \ln\left(1+\exp\left\{\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right\}\right) \\
g'\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right] =& \mathrm{diagrv}\left(\exp\left\{\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right\}\right)\cdot \\
&\cdot\mathrm{diagrv}\left(\mathbf{1}+\exp\left\{\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right\}\right)^{-1}\cdot\mathbf{1}
\end{aligned}
$$

while if we use the second approximation in Equation (16) we have that

$$
\begin{aligned}
g\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right] =& \frac{1}{\varphi}\mathrm{diagrv}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\Phi\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right] + \\
&+\phi\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right], \\
g'\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right] =& \Phi\left[\frac{1}{\varphi}\left(\Lambda_{t|t-1}\cdot\boldsymbol{\beta}_{t|t-1} - r_{LB}\mathbf{1}\right)\right],
\end{aligned}
$$

Opschoor and Van der Wel (2022) point out that the second specification of the function g was inspired by the ZLB forward-rate approximation of Krippner (2015) and Wu and Xia (2016) in the context of a shadow-rate affine term structure model. It is preferred in empirical applications reason for which we will use it as well in our framework.

2.2.4. Model Properties

TO BE FINALISED

3. Empirical Results

This section discusses the estimation implications of applying the models to a bond prices dataset introduced in the first subsection.The second subsection discusses additional model assumptions and restrictions necessary to fit the dataset. The third subsection introduces the in-sample estimation results for the models introduced and the fourth section the out-of-sample forecasts of the individual bond prices series.

3.1. Data

This paper is based on a U.S. Treasuries dataset obtained from Bloomberg. It consists of monthly observations of end of month prices of all coupon bearing U.S. Treasuries and Bills from June 1997 to November 2014, for a total of 204 monthly data points, sorted at each time period by months to maturity. The maturity in months at every observation date is approximated to the closest integer. Then each month in the term structure is considered as 30.4375 days. From these we have removed all callable bonds and all bonds that carried any sort of optionality linked to their coupon or notional. In the case in which the are more bonds available for a given maturity at any time period, the bonds are then selected based on the "highest coupon" criteria since, as noticed by McCulloch (1975) and Schaefer (1981), bonds with lower coupons could be trading on discount and to be more often mis-priced due to the higher tax advantage on their cashflows. This criterion is applied also in case of newly issued bonds which replace old on-the-run bonds with lower coupons. Moreover in order to further avoid liquidity issues as reported by Bliss (1996), Treasury Bill prices are used to represent bonds with 12 or less months to maturity. In this way each individual series in the dataset constructed represents at each point in time the price of an available bond with a given time to maturity. Each time series of this panel will be referred to from now onwards as the constant maturity bond price series for a given maturity.

Looking at Figure .5 the first thing to notice is that at any point in time from 1993 onwards, excluding some small periods between 1999 and 2006, there is always a bond at every consecutive monthly maturity until 5 years. Between 5 and 7 years there are a few missing observation gaps of 2 or 3 months between consecutive bond maturities which tend to become larger between the maturities up to 8 years. Beyond 8 years there are very few bonds available with consecutive maturities. These characteristics of the dataset are due to the US policy of regularly issuing bonds with maturities of 5 and 10 years. The 30 years bond program has been discontinuous up until February 2002, then suspended and slowly reintroduced from February 2006. Despite these few gaps in the data, the US treasury bond dataset is one of the most complete government bonds dataset, much denser compared to the UK or other European countries. For these reasons the dataset considered includes only prices of bonds with maturity up to 7 years, rearranged as constant maturity bonds from 1 to 84 months to maturity.

The characteristics of each bond price series differ widely across the dataset. This is because, given the large time period considered, the economic situation and the market access possibilities for refinancing of the US debt varied considerably. This has been reflected in the size of the coupons, which is larger for long maturity bonds issued in a higher interest rates regimes, and in the liquidity of each individual issuance.

3.2. Bond-FSN Signal Specification and Cointegration Hypothesis

In empirical studies of the term structure it has been often found that interest rates series have a highly persistent behaviour. In the literature rates series have been modelled assuming that their dynamics is driven mostly by two or three common factors[27]. These factors have been identified in level, slope and curvature of the term structure and used to motivate the dynamics of models like the DNS of Diebold and Li (2006). On the other hand, following the seminal work of Anderson et al. (1992), other attempts have modeled persistent rates as a cointegrated system[28]. Given the one-to-one relationship between the discount curve and the spot rates curve it is reasonable to believe that the discount rates share similar features.

[27]See Diebold and Rudebusch (2013) for an extensive discussion on the literature on modelling interest rates.
[28]Dungey et al. (2000), Giese (2008), Jardet et al. (2010).

In the estimation of the dataset presented in Section 3.1, the present paper takes the latter view and restricts the states dynamics of the first model to have a cointegrating relation between discount rates as in the FSN of Bowsher and Meeks (2008). The reason for this approach is motivated in particular by the empirical results of Bowsher and Meeks (2008) and of Giese (2008).

In order to represent this assumption, the dynamic equation of the model is rearranged in terms of common trends setting $\boldsymbol{\Psi} = \boldsymbol{I}$ while restricting $\bar{\boldsymbol{\gamma}}$ and $\boldsymbol{\Pi}$. Assuming one common trend and rearranging the series appropriately the model becomes

$$\mathbf{p}_t = (\mathbf{C}_t + \mathbf{I})\, \mathbf{W}_B \boldsymbol{\gamma}_t + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\gamma}_t = \begin{pmatrix} 0 \\ \bar{\boldsymbol{\alpha}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{\Pi} \end{pmatrix} \cdot \boldsymbol{\alpha}_t + \boldsymbol{\xi}_t$$

$$\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t$$

For the purpose of this paper the degrees of the B-splines in $\mathbf{W}_B$ used is set to be $k = 3$ since it is the smallest degree for the polynomials to have enough flexibility to replicate the term structure curve through a continuous first and second derivative. Higher degrees splines have been considered but for the presented dataset the improvements were not significant enough to justify the increase in dimensionality of the parameter space. The selected knots vector from our selection procedure is of dimension $l = 6$, with positions $\mathbf{k} = \{1, 4, 6, 8, 38, 84\}$, which results in $m = 8$ dynamic spline parameters. Then selecting 2, $m - l = 2$, more arbitrary positions in the cross-section, let's say $\{12, 40\}$[29] we can apply the factor rotation in and the dynamics of the discount function would be now driven by the dynamics of the discount rates with maturities $\{1, 4, 6, 8, 12, 38, 40, 84\}$. The knots selection is performed assuming that the Bond-FSN model has only one common trend set up as the second element of the spline parameters vector. Then the cointegrating assumption can be tested empirically on the subset of discount rates in the dataset. In order to do so the series of the vector $\boldsymbol{\delta}_t$ is constructed using the bootstrapped discount rates from our dataset. Then the Johansen

[29] Any other pair could have been picked, the results are independent from their choice.

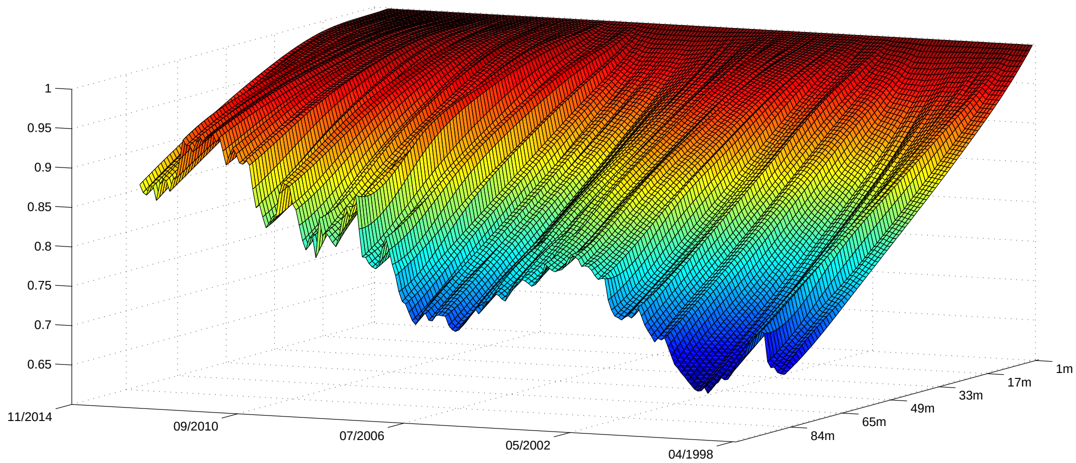

Figure 1: The extracted signal discount function from the Bond FSN model.

(1991) cointegration test is performed. The results are summarized in Table .1.

r is the cointegration rank, the number of cointegrating vectors, which assumes for each test $m-r$ common trends. The critical values are the 0.05 confidence interval values under the H_0 hypothesis.

The null H_0 hypothesis of $r = m-1 = 8$ is not rejected with a quite high p-value implying that the existence of one common trend is highly probable. However, the null hypothesis of cointegration with rank $r = m-2 = 7$ is also not rejected with a reasonably high p-value suggesting that maybe it could be possible to model the discount signal with two common trends.

3.3. In-Sample Term Structure Estimation results

This section assesses the ability of the models to fit the observed bonds prices surface and extract the term structure of interest rates. In Figure 1 it is possible to see that the extracted discount signal function by the one common trend Bond FSN model is smooth, bounded by 0 and 1 and almost everywhere decreasing with maturity, apart from few endpoints. The reason for this behaviour is strongly dependent on the number and positions of the spline knots in relation to the maturities of the missing bonds at each time step. However the flexible structure of the B-Splines allows for a clear representation of the estimated term structure of interest rates over time consistent in most of the cases with the non arbitrage theory. However the Kalman Filter doesn't allow to impose smoothness restrictions which would guarantee the presence of positive forward rates or a zero lowerbound.

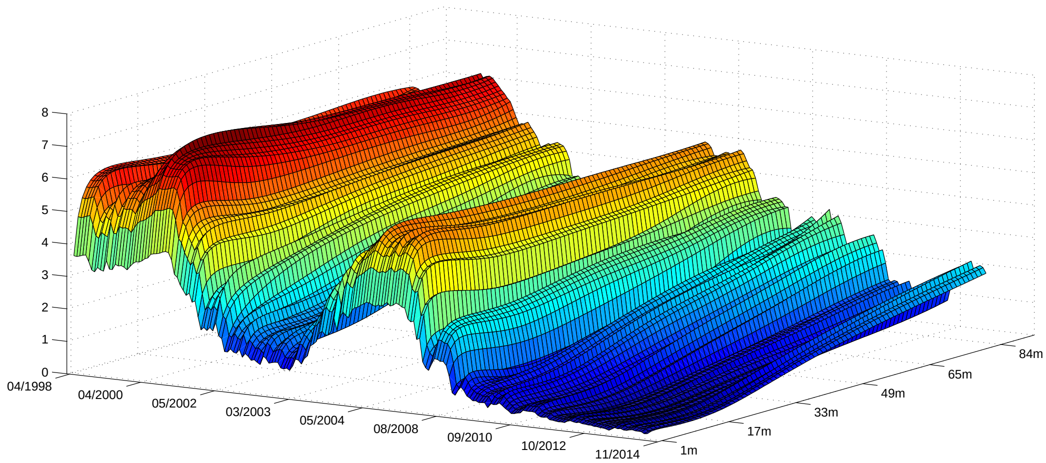

Figure 2: The extracted term structure from the Bond FSN model.

The term structure curves are almost everywhere increasing apart from small flat-decreasing behaviours in time periods which coincide with a slowdown of the US economy. As can be seen in Figure 3, the implied forward rate curves are everywhere positive without giving any signal of over-smoothing. Moreover the forward curve is also positive in the most recent time periods where the interest rates in the term structure, particularly in the short end, tend to be very close to 0. The forward rate curves are almost everywhere above the spot curves exceptions for the time periods when the spot curves becomes flat and at some maturities where the forward rates tend to cross the spot rates curves.

On the other hand we have fitted on the dataset all the presented specifications of the Bond-DNSS model, which include both time invariant and time varying scale $h_{t|t-1}$ as well as $\lambda_{it|t-1}$, with or without a zero lowerbound restrictions and assuming both a Gaussian and student t distribution for the price vector $\mathbf{p}_t$. The results are presented in table Table .2

From an in-sample fit comparison the best models at fitting the data are the one which include the robustifying assumption of students t distributed bond prices. The preferred specifications includes only $h_{t|t-1}$ for both the distribution with a slight preference towards the models which don't include the zero-lowerbound restriction. The term structure extracted by the Bond-DNSS model are consistent with the fit of the static DNSS; Figure .6 and Figure .8. The fitted $h_{t|t-1}$ time heteroscedastic parameter is quite persistent with large fluctuations across time; Figure .9. This suggests that the simplistic cross-sectional heteroscedastic assumption is not enough to capture fully the heteroscedasticity in the bond

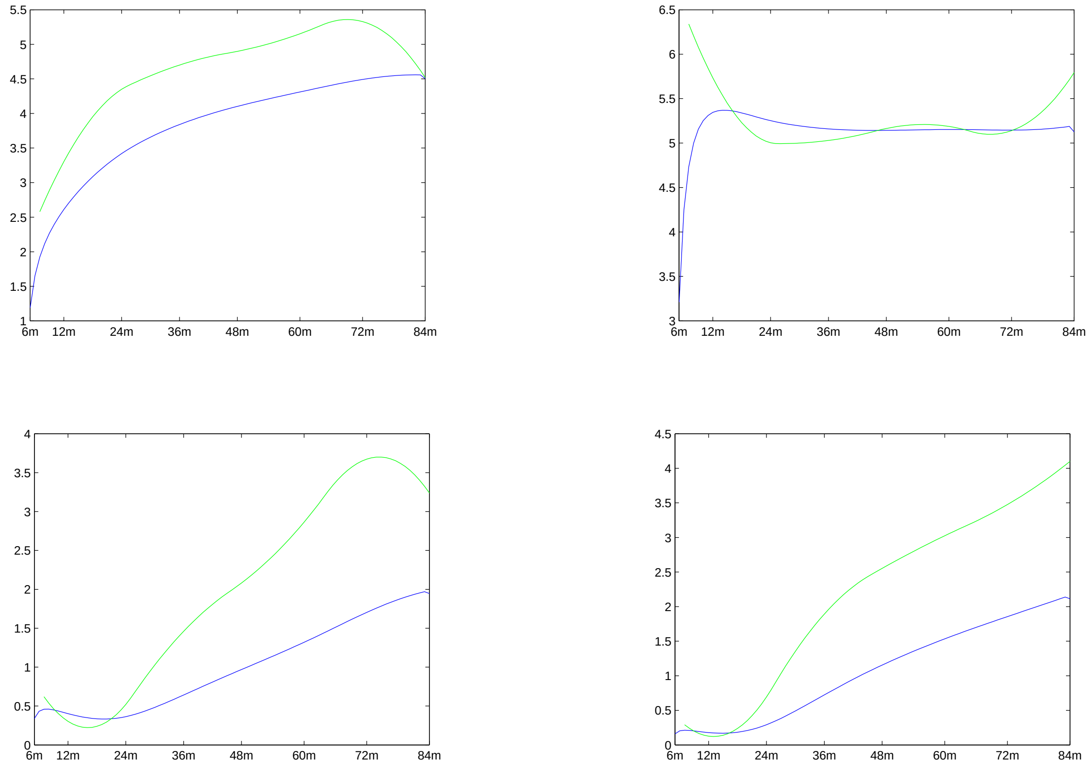

Figure 3: The spot rates (Blue Line) and forward rates (Green Line) curves on the 07/2002 (Top Right), 07/2006 (Top Left), 10/2010 (Bottom Right), 05/2014 (Bottom Left).

prices measurement error. At last the time varying λ_i parameters allow for a wider flexibility in the time variation of the curves. However they are mostly not persistent with larger variation only around the 2008 periods and don't contribute significantly to the improvement of the fit in this case; Figure .10.

3.3.1. In-Sample Bond Prices Estimation results

The in-sample fit of the models is ultimately assessed looking at how can accurately fit the bond prices surface. The performance of the Bond FSN model and of the Bond-DNSS are then compared against the in-sample fit obtained after extracting the term structure curves with the duration weighted NSS model, as estimated by Gurkaynak et al. (2007), and with a duration weighted static B-Spline model[30]. In fitting the B-Splines to the unobserved discount curves, we have used the same knots positions of the Bond FSN model.

[30] A B-Spline model to extract the term structure which fits the cross section of bonds with the same criteria used by Gurkaynak et al. (2007) for the NSS model: Through WLS assuming that the weights are the inverse of the durations of the bonds in the cross section at each t.

Two specification for the Bond FSN model have been considered: a two common trends specification, and a standard dynamic factor structure for the rates in levels with three unobserved factors where, following Jungbacker et al. (2014), we have imposed as identification restriction that the rows $\{1, 5, 9\}$ of the $\mathbf{\Pi}$ matrix in (5) are replaced with the vectors $\{(1, 1, 0)', (1, 1/2, 1)', (1, 0, 0)'\}'$. In this way the three unobserved factors can be related to the level, slope and curvature factors described in the term structure literature.

The comparison is then performed looking at the bond prices Mean Squared fitted Errors from each of the models. In doing so we have used the approach of Bliss (1996) which suggest to weight the Mean Squared Error from the fit by the inverse of the duration of the bonds. These Weighted Mean Squared fitted Errors (WMSE) allows for an easier comparison free of other bond specific characteristics which could affect the fit of each models.

In Figure .11 we show how the WMSE for the fit of each of the models varies across the time dimension of the dataset in proportion to all the others. This has been averaged for all the models across three portions of the bonds maturity cross-section: Between 1 month and 28 months (Short End), between 29 months and 56 months (Mid) and 57 months and 84 months (Long End). In all the three cases the worst performers are the NSS models and the Bond FSN with Dynamic Factors, while on the other hand the fit of the static B-Spline model and of the two Bond FSN with common trends models are comparable. In particular these three model outperform significantly the other two at the short and long maturities. At the shorter maturities bond prices are the result of averaging across only few rates, e can see that the three models clearly better fit the short end of the term structure. At the mid maturities the fit is roughly comparable among all the models a part from, in some periods, a preference towards the static B-Spline models.

Another comparison can be made looking at the fit of the individual bond prices. As a reminder, given the setup of the model presented in this paper, at each point in time t a treasury bond P^j with coupon c_j enters the dataset the first time when it has a maturity of $84m$. It enters the vector $\mathbf{p}_t \in \mathbb{R}^{82}$ in the last position with $\tau = 84$, then at every sequent time step $t + 1$ it would switch to the position $\tau + 1$ until it gets removed when either it reaches $\tau = 13$ or another bond appears in the dataset with the same maturity but a higher coupon. Then for a bond p_t^j with maturity τ at time t, for $1 \leq \tau \leq 84$, its fitted price can

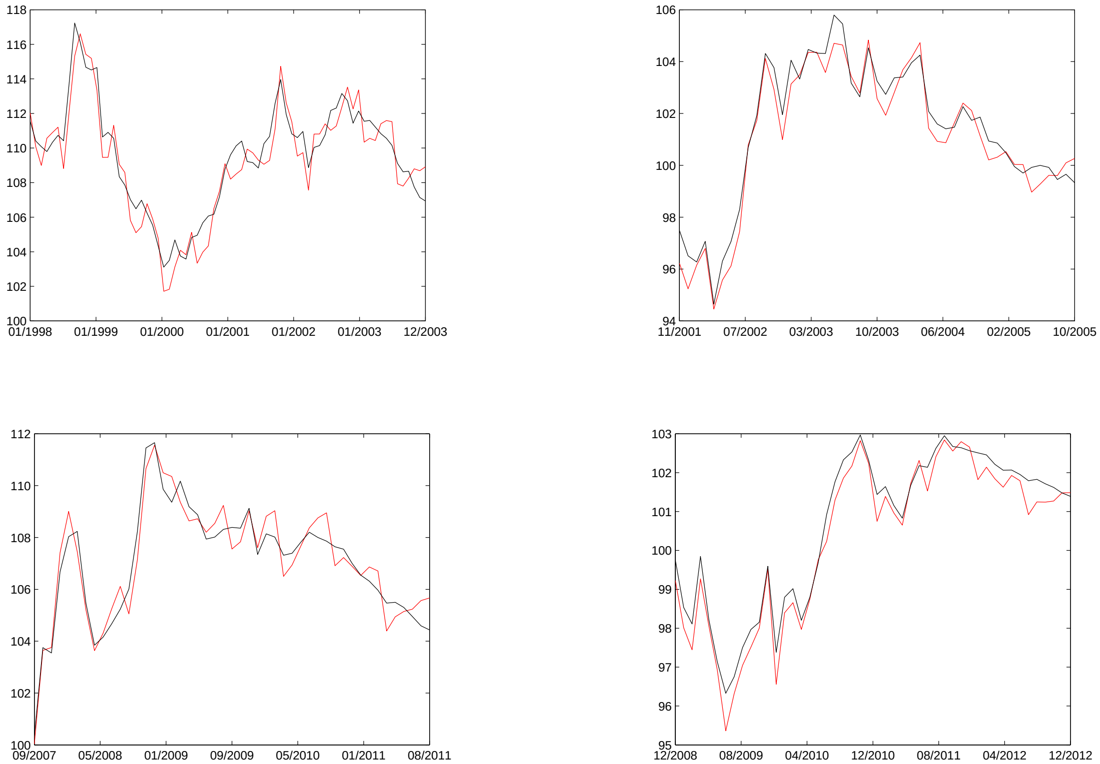

Figure 4: Observed price series (black line) and Fitted price series from the Gaussian FSN model (red line) of the Treasury 7 1/2 02/15/05 between 01/98 and 12/03 (Top Left), the Treasury 3 1/2 11/15/06 between 11/01 and 09/05 (Top Right), the Treasury 4 1/4 09/30/12 between 09/07 and 08/11 (Bottom Left), the Treasury 1 1/2 12/31/13 between 12/08 and 11/12 (Bottom Left).

be described as

$$\hat{P}^j_{\tau t} = \hat{P}_{\tau t} = \frac{c_j}{2} \sum_{i=0}^{\lfloor \tau/6 \rfloor} \hat{\delta}_{(\tau - i6)t} + 100\hat{\delta}_{\tau t}$$

where each $\hat{\delta}_{\tau t} = \sum_{q=1}^{m} \omega^B_{\tau q} \hat{\gamma}_{qt}$ and $\hat{\gamma}_{qt}$ are the estimated unobserved factors from the state updating equation of the Kalman Filter. Now at each $t+1$ until maturity the fitted price of the bond will be $P^j_{\tau\, t+1} = P_{\tau - 1\, t+1}$.[31] In Table .3 is presented the average WMSE from fitting a random sample of 24 bonds from the dataset with the five models as a percentage in respect to the NSS model. Also in this case the B-Spline static model and the Common Trend

[31] It is also possible to use the smoothed factors from the Kalman Filter smoothing algorithm instead, which uses all the information from the observations ahead of the smoothed state. However the usage of the updated curves instead reflect how the one-step-ahead forecasts of the Kalman Filter get updated on the day when the new bond prices observations gets released, therefore giving a more realistic indicator of how the model would perform in a real life setting.

Bonds FSN outperform the NSS in almost all the cases. The Bond FSN Dynamic Factor model is the worse performer, particularly with bonds in thee earlier part of the dataset. Half of the times either one of the two common trend Bond FSN model and Bond-DNSS outperforms significantly all the competing models, however the other half of the times they are outperformed by the B-Splines model. The one and two common trends specifications are very close with each other with the one common trends outperforming the two in few cases. This suggests that probably part of the quality in the fit is given away giving more structure to the dynamics.

These results confirm the studies of Steeley (2008) and Bliss (1996) in which it is showed that the B-splines fits the bond prices better. The interpretation for this might be that, as showed by Svensson (1994, 1995), the static NSS has a smooth specification that tries to find a compromise across all the maturities of the term structure while B-Splines having a more flexible specification which, without the correct restriction in the time dynamics, tend to accommodate more for the presence of bond with abnormal prices or coupons. However, as can be seen by the shape of the signal extracted term structure, a good compromise of both can be found in the discount function fitted by the FSN model which smooth out this effect taking into account of the information given by the time dimension of the bond price surface.

3.4. Out-of-sample Forecasting

Given the dynamic specification of the Bond FSN model it is possible to use it to make forecasts of both the term structure curve and the observed bond prices. In this context the one-step ahead forecasts of the price of the bond j from the Bond FSN model is then given by

$$\hat{p}^{j}_{\tau-h\,t+h|t} = \frac{c_j}{2} \sum_{i=0}^{\lfloor(\tau-h)/6\rfloor} \sum_{q=1}^{m} \omega^{B}_{(\tau-h-i6)\,q} \hat{\gamma}_{q\,t+h|t} + 100 \sum_{q=1}^{m} \omega^{B}_{\tau-h\,q} \hat{\gamma}_{q\,t+h|t}$$

where each $\hat{\gamma}_{q\,t+1|t}$ is the one step ahead forecasts of the unobserved factors. In the same way is possible to obtain also multi-step forecasts

$$\begin{aligned}\hat{p}^{j}_{\tau-h\,t+h|t} &= \frac{c_j}{2}\sum_{i=0}^{\lfloor(\tau-h)/6\rfloor}\exp\left\{-\left(\tau-h-i6\right)\boldsymbol{\lambda}_{(\tau-h-i6)}\hat{\boldsymbol{\beta}}_{t+h|t}\right\}+\\ &+100\exp\left\{-\left(\tau-h\right)\boldsymbol{\lambda}_{(\tau-h)}\hat{\boldsymbol{\beta}}_{t+h|t}\right\}\end{aligned}$$

For the Bond FSN Common Trends model the one-step-ahead and the multi-step-ahead forecasts are obtained through the state forecasting equations of the Kalman filter. In order to asses the reliability of these forecasts the forecasting power of the model is compared against other existing models. In this comparison we will drop the dynamic factor Bond FSN model given its poor fitting performance. The first competitor is the the popular DNS model, as introduced by Diebold and Li (2006),

$$\begin{aligned}\mathbf{y}_t &= \boldsymbol{\omega}_y+\Lambda_{ns}\widetilde{\mathbf{f}}_t+\boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \overset{iid}{\sim} N\left(0,\,H\right)\\ \widetilde{\mathbf{f}}_{t+1} &= \boldsymbol{\omega}_f+\Phi\widetilde{\mathbf{f}}_t+\boldsymbol{\eta}_{t+1} \quad \boldsymbol{\eta}_t \overset{iid}{\sim} N\left(0,\,Q\right)\end{aligned}$$

this model is a generalization of the static NS specification where the three static parameters for level, slope and curvature are substituted by three dynamic factors $\widetilde{\mathbf{f}}_t=\left(\widetilde{f}_{Lt},\,\widetilde{f}_{St},\,\widetilde{f}_{Ct}\right)'$. As for the static model, the matrix of factor loadings $\boldsymbol{\Lambda}_{ns}=\left(\boldsymbol{\lambda}^1_{ns}\left(\tau_1\right),\,\boldsymbol{\lambda}^2_{ns}\left(\tau_2\right),\,\boldsymbol{\lambda}^3_{ns}\left(\tau_3\right),\,...,\,\boldsymbol{\lambda}^N_{ns}\left(\tau_N\right)\right)'$, where $\boldsymbol{\lambda}^i_{ns}\left(\tau_i\right)=\left(1,\,\frac{1-\exp(-\lambda\tau_i)}{\lambda\tau_i},\,\frac{1-\exp(-\lambda\tau_i)}{\lambda\tau_i}-\exp\left(-\lambda\tau_i\right)\right)'$, is fixed and only depends on the rates maturities τ_i and the parameter λ that gets calibrated[32]. The second one is the original FSN-ECM presented by Bowsher and Meeks (2008),

$$\begin{aligned}\mathbf{y}_t &= \boldsymbol{\omega}_y+\boldsymbol{\Lambda}_{\boldsymbol{bm}}\bar{\mathbf{f}}_t+\boldsymbol{\varepsilon}_t \qquad \boldsymbol{\varepsilon}_t \overset{iid}{\sim} N\left(0,\,H\right)\\ \Delta\bar{\mathbf{f}}_{t+1} &= \boldsymbol{\alpha}\cdot\left(\boldsymbol{\beta}'\cdot\bar{\mathbf{f}}_t-\boldsymbol{\mu}\right)+\boldsymbol{\Psi}\cdot\Delta\bar{\mathbf{f}}_t+\boldsymbol{\eta}_t \quad \boldsymbol{\eta}_t \overset{iid}{\sim} N\left(0,\,Q\right)\end{aligned}$$

This is a dynamic factor model which targets directly the rates curves with a spline function. The factor loadings described by the matrix $\boldsymbol{\Lambda}_{bm}=\mathbf{W}$ are fixed spline coefficients as described in Poirier (1973) and Monahan (2001). Also in this case the factors $\bar{\mathbf{f}}_t$ are modelled as they would have been a subset of the rates on spot curve $\mathbf{y}_t$ at the knots maturities. Their

[32]For a discussion on the calibration of λ please see Diebold and Rudebusch (2013).

dynamics is then driven by an error correction model (ECM) between the rates at the knots of the spline. The last model considered is a simple Random Walk (RW) which assumes that the forecasts are simply equal to the last fitted values.

Considering that the Bond FSN and the Bond-DNSS models presented here are the first of its kind used to model directly the bond prices while the other existing models introduced are designed to model only previously extracted term structure curves, the comparison is then performed applying all the existing dynamic models on both the NSS and B-Splines term structure samples extracted for the in-sample comparison. Once the dynamics models are fitted to the samples, they are used to produce one, six and twelve step ahead forecasts of the rates curves. These curves are then converted in discount curves and used to reconstruct the fitted prices of the bonds which are then compared with the observed bond prices. In this preliminary forecasting exercise the Bond-NSS specification which we are considering is the one which assumes a students t distribution and a time varying scale component $h_{t|t-1}$.

3.4.1. Bond Prices Forecasting Results

For the forecasting comparison we have both analysed the average forecast power of the models across groups of bonds with different maturities (Low, Mid, High) and assessed the individual forecasting power on the same selection of bonds used in the in-sample comparison. We have focused on two arbitrary $H = 24$ months periods, from the 03/2004 and the 02/2006 (Period A), as well as from the 12/2012 and the 11/2014 (Period B). The forecasts are made in a pseudo out-of-sample fashion and, as in the in-sample case, the comparison is performed by looking at the root mean squared forecast errors weighted by the duration of the bond at that point in time. Moreover the significance of the comparison for individual bond series is also assessed looking at the results of the Diebold-Mariano (D-M) test for superior forecasting performance, which results are only used to highlight the cases in which the forecasting performance is statistically significant at a 5% confidence level. Forecasts are compared estimating recursively the model using all the data up to $t - h + 1$, with t being either one of the two sample termination date 01/2004 or 10/2012, then re-estimating the parameters at each new time step.

In Tables .4, .6 and .7 the forecasts for the five competing models are performed using the B-Splines term structure dataset as a sample while the forecasts in Tables .5, .8 and .9 are performed using the NSS sample. From this it appears that the performance of all the three models DNS, FSN-ECM and RW is very different between the two sets of tables. This confirms the fact that the forecasting performance will be dependent on which term structure sample is used, and they seems to performs better when estimated using the B-Splines term structure sample, with the large differences for longer forecasts and in Period A. This matches the fact that the B-Splines dataset, compared to the NSS dataset, better fit the bond price surface, and implies that the forecast accuracy improves in existing dynamic term structure models while using a more accurate fitted term structure sample.

The relative performance of the DNS and FSN-ECM varies across the forecast length and subsamples considered. As can be seen from Tables .4 and .5 the FSN-ECM outperforms on average the DNS at all the maturities and at all forecasting horizon. At the individual bonds level, on average the FSN-ECM outperforms the DNS in Period B at shorter forecasting horizons but is outperformed at longer forecasting horizons, significantly while using the B-Splines sample in Table .6. However in Period B the situation is reversed significantly both while using the B-Splines and NSS sample in Tables .7 and .9 respectively.

The Bond FSN and Bond-DNSS models, while their forecasting performances are unrelated to any extraction sample used, on average they consistently outperforms all the competing models at every maturity groups in comparison with the DNS regardless of the extraction sample considered, Tables .4 and .5. Across the individual bonds series it outperforms the DNS model significantly mostly in Period B, as can be seen from Tables .7 and .9 at both 1 and 12 months forecasting horizon. However it is also outperformed significantly in few cases at a 6 month horizon in Period A while the DNS is estimated with a B-Spline sample, Table .6.

Looking at the relative performance of two common trends Bond FSN and the Bond-DNSS the performance of the two models is similar, however on average the Bond-DNSS model slightly outperforms the Bond FSN, Tables .4 and .5. On the other hand, looking at the individual bonds series there is not clear cut between the two. Sometimes the two components tends to outperform more the one component in case of longer maturity bonds. This can suggests that the second common trends can help to capture the behaviour of longer rates giving a higher degree of flexibility to forecast longer maturity bonds. In any case the one common trend model still has a satisfactory performance which outperforms all the existing competitors, significantly in most of the cases.

4. Conclusion

The present study introduces two dynamic models for modelling the unobserved term structure of interest rates from observed bond prices. The aim is a direct improvement on the forecasting accuracy of the observed bond prices when modelled with a dynamic term structure in comparison with existing models, which forecasting accuracy depends on the fit of the previously extracted interest rates samples.

The first model generalises the FSN framework, by defined by Bowsher (2004), modelling the unobserved term structure curve in the form of a discount function with B-Splines, in order to better capture the term structure stylized features with fewer restrictions. The factor dynamics of the term structure signal is driven by a cointegrating relation between the discount rates on the spline. The number and location of spline knots is defined by a knots selection algorithm based on an information criterion and a formal statistical tests. The cross-sectional heteroscedasticity is assumed to be deterministically time-varying and proportional to the bond duration at every point in time. Through modelling directly the discount functions, the unobserved dynamic factors are linear in bond prices allowing the use of exact filtering techniques for signal extraction and forecasting. This provides a fast combined framework for both the extraction and the forecasting of the term structure curves and the observed bond price surface in one single estimation step.

The second model exploits the non linearity in the bond prices measurement errors assuming to be multivariate t distributed, while the unobserved discount function signal has a NSS factor structure. The factors are then modelled dynamically with a score-driven specification. Moreover, the other two shape parameters of the NSS specification are also allowed to vary over time providing more heterogeneity to the NSS specification to capture the shape of the term structure over time. In addition, while the cross-sectional heteroscedasticity is captured in the same way as in the first model, an additional time varying parameter is used to model the time variation in the time heteroscedasticity of bond prices.

For the second model we study the model properties and derive the conditions for stationarity and invertibility of the score-driven filter.

From an in-sample estimation prospective, the models fit bond prices time series outperforming other static term structure extraction models, like the static Nelson-Siegel model, while providing reasonable estimates of the signal term structure which follow closely the features of the theoretical term structure. An out-of-sample forecasting comparison with popular term structure dynamic models is provided which includes models such as: the DNS and the FSN-ECM of Bowsher and Meeks (2008). This study highlights the fact that that the accuracy of these models in forecasting observed bond prices is strongly dependent on how accurately the term structure sample used for the estimation of the parameters it has been extracted. Moreover in forecasting the bond prices surface, the model introduced outperforms all the considered existing models, significantly at the one and twelve months ahead horizons.

Through this comparison this study aims to highlight that since the term structure of interest rates is an unobserved object entirely dependent on the observed traded bond prices, it is important to reconcile these two while making forecasts. This is is because if the term structure forecasting step is completely unrelated to the extraction step, and the two steps are performed separately using techniques with different assumptions and aims, it is difficult to track the propagation of the fitting errors in the forecasts. However we have shown that there is a benefit in the forecasting performance in both fitting and forecasting the term structure in the same inferential framework bringing these two steps together and using all the information available in the bond prices surface at every point in time.

In conclusion, in addition to all the aforementioned features, the models here presented can accommodate also for various extensions. Due to its linearity in the factors, the unobserved components structure can also allow for the separate inclusion of additional linear unobserved components as additional signals that can be used to directly describe other different dynamic features of bond prices and of the term structure dynamics, like liquidity premia, term premia, credit spreads, breakeven inflation, etc. Moreover, the model can accommodate also for dynamics factors driven by exogenous explanatory variables, like macro variables as in the Global DNSS introduced by Diebold and Rudebusch (2013). On the same line, the state space structure also allows for the inclusion in the transition equation of other observed time series which are based on the same unobserved term structure signal function, like CDS series or Bond indexes.

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Figures

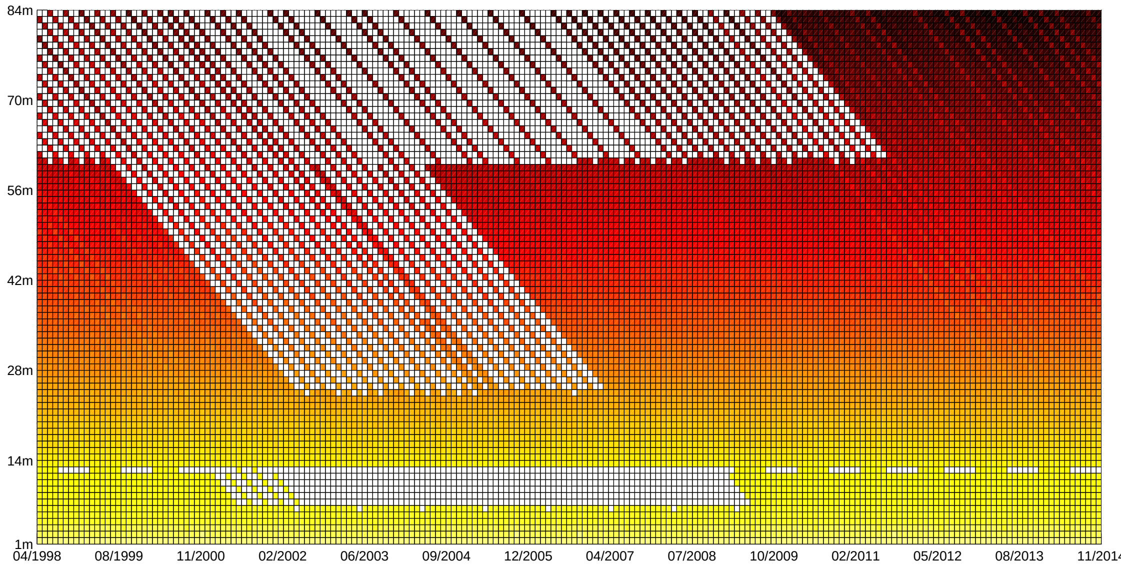

Figure .5: Bond Prices availability per dates and respective durations, darker the colour higher the duration.

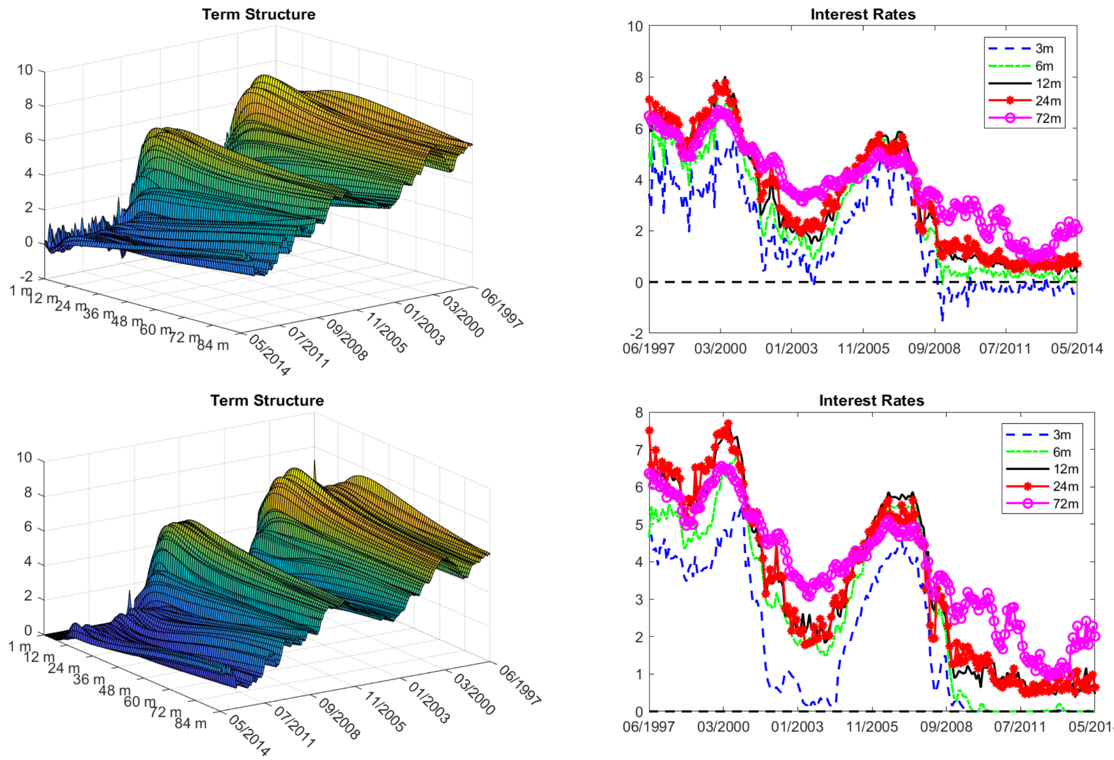

Figure .6: The extracted term structure with the t distributed Bond-DNSS model with time varying scale $h_{t|t-1}$, without (Top) and with zero-lowerbound (Bottom).

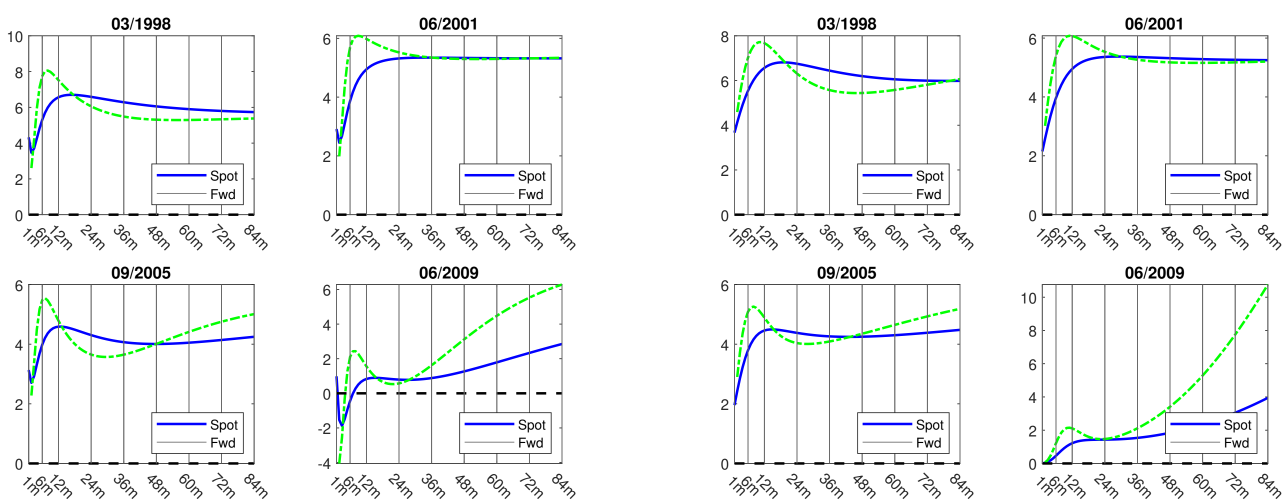

Figure .7: Spot and forward rates curves from the t distributed Bond-DNSS model with time varying scale $h_{t|t-1}$, without (Left) and with zero-lowerbound (right).

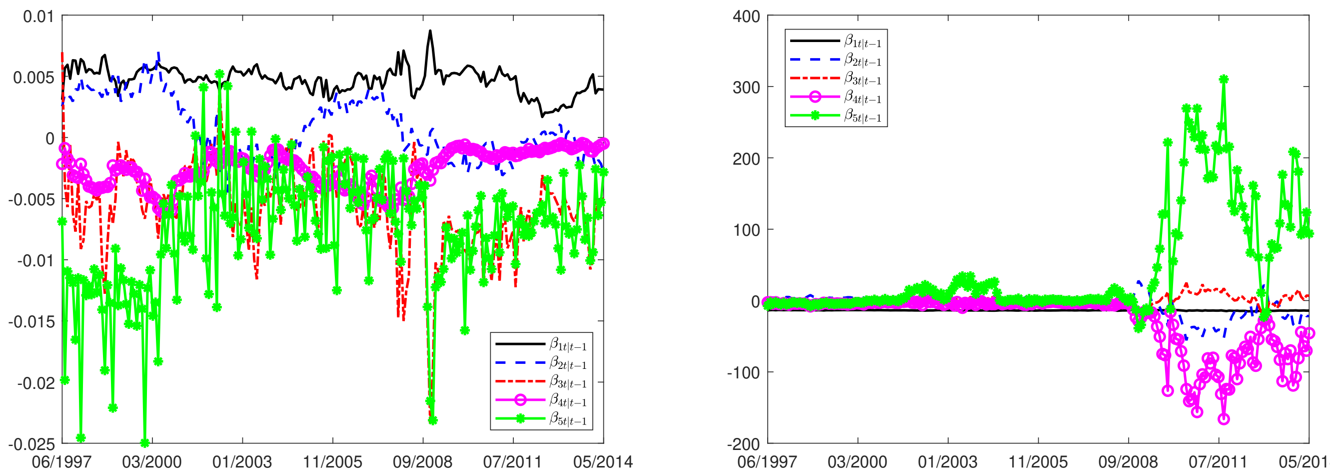

Figure .8: NSS time varying factors $\boldsymbol{\beta}_{t|t-1}$ from the t distributed Bond-DNSS model with time varying scale $h_{t|t-1}$, without (Left) and with zero-lowerbound (right).

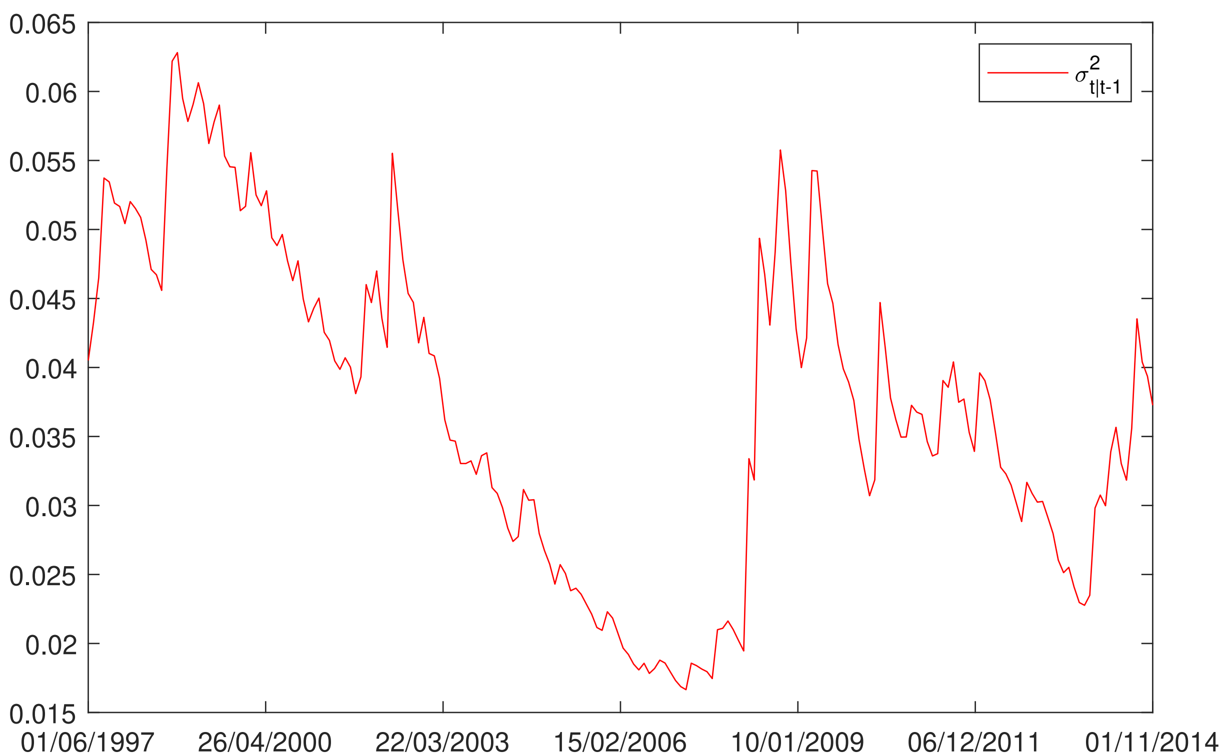

Figure .9: Plot of the filtered $\sigma^2_{t|t-1} = \exp\left(h_{t|t-1}\right)$ time varying heteroscedastic component of the Bond-DNSS.

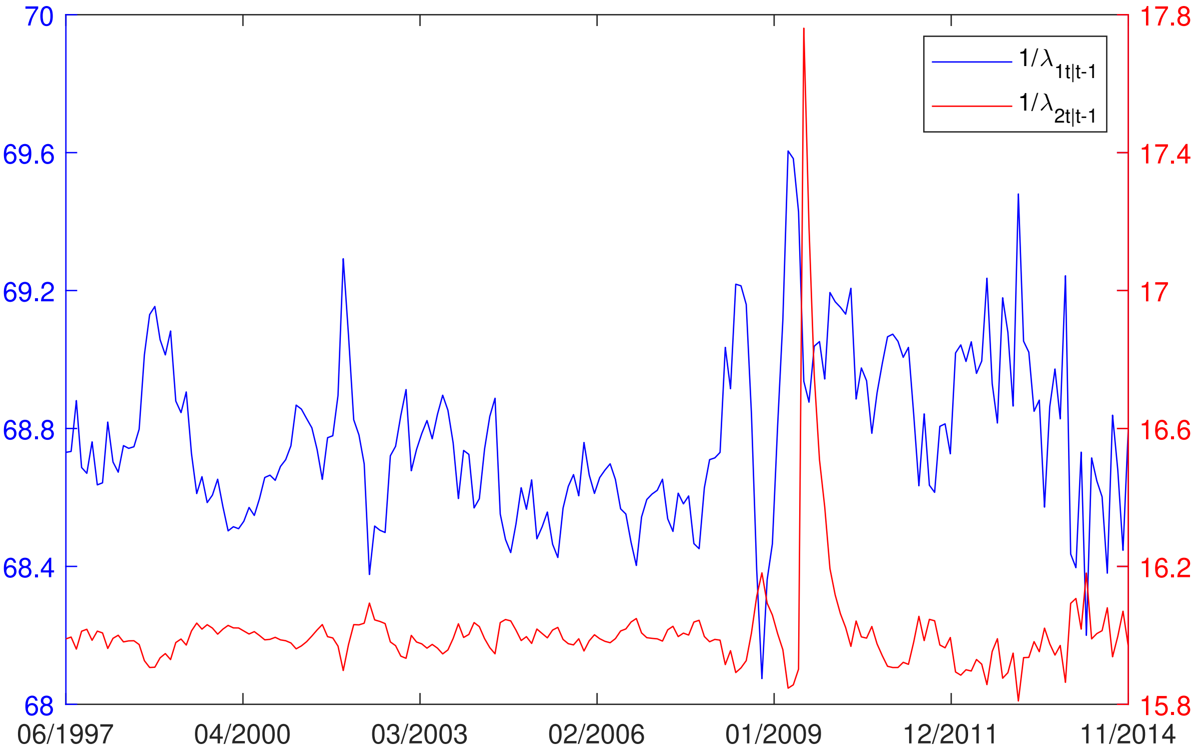

Figure .10: Plot of the filtered $1/\lambda_{1t|t-1}$ and $1/\lambda_{2t|t-1}$ shape parameters for the term structure specification of the Bond-DNSS.

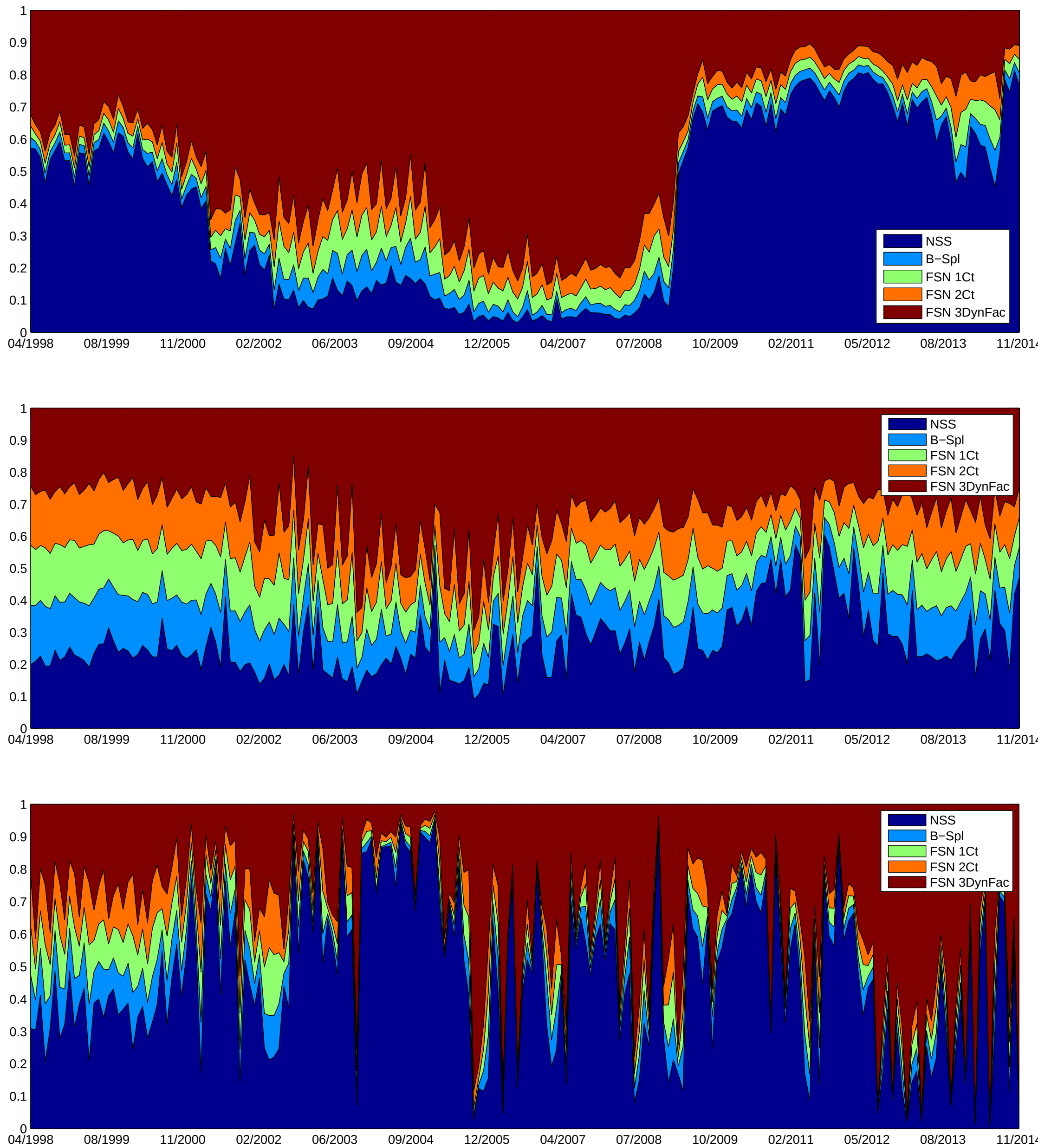

Figure .11: Average WMSE by bond maturities: for short bond maturities between 1m and 28m (Top), for mid bond maturities between 29m and 58m (Mid), for long bond maturities 59m and 84m (Bottom).

Tables

r	Johansen Statistic	Critical Values	p-value
0	637,98	203,75	0,0000
1	491,28	167,48	0,0000
2	364,49	134,68	0,0010
3	226,55	103,85	0,0010
4	133,54	76,97	0,0010
5	78,39	54,08	0,0010
6	42,09	35,19	0,0081
7	12,53	20,26	0,4462
8	2,70	9,16	0,6777

Table .1: Johansen Cointegration Test

	Gaussian						Students t					
$\boldsymbol{\beta}_{t\|t-1}$	x	x	x	x	x	x	x	x	x	x	x	x
$h_{t\|t-1}$		x	x		x	x		x	x		x	x
$\lambda_{it\|t-1}$			x			x			x			x
Zero Lowerbound				x	x	x				x	x	x
AIC	60,244.53	**48,771.59**	51,004.31	49,318.53	**48,594.81**	48,976.30	61,434.58	**47,379.43**	57,692.98	51,681.00	**47,297.00**	49,459.00
BIC	60,384.01	**48,926.57**	51,190.28	49,465.76	**48,757.54**	49,170.02	61,581.81	**47,542.16**	57,886.70	51,835.98	**47,467.48**	49,660.47
Logl	- 30,104.26	**- 24,365.80**	- 25,478.16	-24,640.27	**- 24,276.40**	- 24,463.15	- 30,698.29	**-23,668.72**	- 28,821.49	- 25,820.50	**- 23,626.50**	-24,703.50

Table .2: Goodness of fit of all the specifications of the Bond-DNSS model.

Bond	Maturity (Months)	NSS	B-Spline	Bond FSN 2 Ct	Bond-DNSS	Bond FSN 3 Fact
T 6 5/8 05/15/07	69m	1,000	0,982	0,931	**0,911**	3,654
T 6 1/8 08/15/07	66m	1,000	**0,639**	0,884	0,922	3,807
T 5 1/2 02/15/08	60m	1,000	**0,574**	0,830	0,841	4,647
T 5 5/8 05/15/08	57m	1,000	**0,517**	0,574	0,626	2,448
T 3 1/8 09/15/08	30m	1,000	**0,656**	0,661	0,725	4,183
T 4 3/4 11/15/08	51m	1,000	**0,178**	0,298	0,321	1,076
T 3 3/8 12/15/08	27m	1,000	**0,521**	0,631	0,659	1,995
T 3 1/4 01/15/09	26m	1,000	0,696	0,666	**0,657**	1,878
T 2 5/8 03/15/09	24m	1,000	**0,772**	0,794	0,797	1,529
T 5 1/2 05/15/09	45m	1,000	**0,172**	0,252	0,282	0,612
T 6 08/15/09	42m	1,000	**0,129**	0,157	0,156	0,335
T 3 3/8 09/15/09	18m	1,000	0,917	**0,912**	0,962	2,026
T 1 7/8 06/30/15	58m	1,000	0,272	**0,215**	0,222	1,071
T 1 3/8 11/30/15	43m	1,000	0,217	0,204	**0,195**	0,881
T 2 1/8 12/31/15	42m	1,000	0,192	0,182	**0,174**	0,921
T 3 1/4 05/31/16	59m	1,000	**0,142**	0,158	0,153	0,696
T 4 7/8 08/15/16	57m	1,000	0,246	**0,196**	0,202	0,713
T 2 3/4 11/30/16	53m	1,000	0,104	0,113	**0,104**	0,547
T 4 5/8 02/15/17	51m	1,000	0,165	**0,149**	0,150	0,464
T 3 02/28/17	50m	1,000	0,106	0,105	**0,103**	0,437
T 8 7/8 08/15/17	45m	1,000	**0,430**	0,453	0,460	0,698
T 9 1/8 05/15/18	36m	1,000	**0,441**	0,448	0,465	0,763
T 4 08/15/18	33m	1,000	0,272	0,250	**0,244**	1,170
T 8 7/8 02/15/19	27m	1,000	0,576	0,552	**0,532**	0,991

Table .3: WMSE for some selected bond prices series.

AWRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
02/2004-02/2006															
Short	0,246	0,216	0,290	0,214	0,233	0,253	0,242	0,287	0,235	0,238	0,264	0,282	0,276	0,272	0,265
	(1,000)	(0,877)	(1,179)	(0,867)	(0,946)	(1,000)	(0,959)	(1,136)	(0,928)	(0,943)	(1,000)	(1,069)	(1,047)	(1,032)	(1,003)
Medium	0,221	0,159	0,259	0,160	0,164	0,228	0,188	0,273	0,171	0,174	0,237	0,228	0,269	0,215	0,206
	(1,000)	(0,717)	(1,170)	(0,726)	(0,744)	(1,000)	(0,824)	(1,197)	(0,749)	(0,761)	(1,000)	(0,964)	(1,138)	(0,907)	(0,871)
Long	0,154	0,081	0,131	0,092	0,082	0,157	0,096	0,133	0,088	0,083	0,164	0,113	0,138	0,104	0,100
	(1,000)	(0,528)	(0,851)	(0,598)	(0,534)	(1,000)	(0,611)	(0,844)	(0,562)	(0,527)	(1,000)	(0,692)	(0,842)	(0,635)	(0,614)
11/2012-11/2014															
Short	0,210	0,174	0,168	0,176	0,185	0,216	0,174	0,168	0,173	0,181	0,234	0,177	0,171	0,173	0,180
	(1,000)	(0,827)	(0,799)	(0,839)	(0,881)	(1,000)	(0,805)	(0,776)	(0,800)	(0,836)	(1,000)	(0,759)	(0,730)	(0,740)	(0,771)
Medium	0,241	0,138	0,143	0,164	0,157	0,260	0,144	0,140	0,152	0,144	0,295	0,155	0,164	0,160	0,157
	(1,000)	(0,571)	(0,592)	(0,681)	(0,652)	(1,000)	(0,554)	(0,539)	(0,586)	(0,552)	(1,000)	(0,527)	(0,555)	(0,542)	(0,532)
Long	0,188	0,085	0,115	0,142	0,126	0,208	0,105	0,107	0,131	0,117	0,245	0,131	0,167	0,144	0,137
	(1,000)	(0,455)	(0,613)	(0,756)	(0,669)	(1,000)	(0,502)	(0,513)	(0,631)	(0,563)	(1,000)	(0,535)	(0,683)	(0,587)	(0,560)

Table .4: 24m periods Average WMSFE comparison of bond prices sorted across maturities groups, when the forecast of existing models is based on a term structure dataset extracted with B-Splines. The percentage comparison respect to the DNS model forecasts is showed in parenthesis.

AWRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
02/2004-02/2006															
Short	0,283	0,239	0,292	0,214	0,233	0,301	0,254	0,291	0,235	0,238	0,301	0,258	0,286	0,272	0,265
	(1,000)	(0,844)	(1,033)	(0,755)	(0,824)	(1,000)	(0,843)	(0,967)	(0,780)	(0,792)	(1,000)	(0,858)	(0,952)	(0,905)	(0,880)
Medium	0,242	0,162	0,270	0,160	0,164	0,263	0,186	0,263	0,171	0,174	0,270	0,200	0,261	0,215	0,206
	(1,000)	(0,669)	(1,117)	(0,664)	(0,681)	(1,000)	(0,708)	(1,000)	(0,650)	(0,661)	(1,000)	(0,743)	(0,968)	(0,795)	(0,764)
Long	0,172	0,102	0,192	0,092	0,082	0,181	0,117	0,191	0,088	0,083	0,193	0,132	0,192	0,104	0,100
	(1,000)	(0,596)	(1,116)	(0,535)	(0,478)	(1,000)	(0,648)	(1,057)	(0,488)	(0,458)	(1,000)	(0,684)	(0,994)	(0,538)	(0,520)
11/2012-11/2014															
Short	0,285	0,213	0,208	0,176	0,185	0,291	0,214	0,211	0,173	0,181	0,303	0,213	0,241	0,173	0,180
	(1,000)	(0,749)	(0,730)	(0,619)	(0,651)	(1,000)	(0,737)	(0,725)	(0,596)	(0,623)	(1,000)	(0,704)	(0,795)	(0,571)	(0,595)
Medium	0,287	0,160	0,163	0,164	0,157	0,300	0,170	0,164	0,152	0,144	0,325	0,173	0,219	0,160	0,157
	(1,000)	(0,555)	(0,566)	(0,570)	(0,546)	(1,000)	(0,567)	(0,546)	(0,508)	(0,478)	(1,000)	(0,531)	(0,672)	(0,490)	(0,482)
Long	0,221	0,133	0,165	0,142	0,126	0,233	0,157	0,156	0,131	0,117	0,258	0,165	0,173	0,144	0,137
	(1,000)	(0,604)	(0,750)	(0,643)	(0,569)	(1,000)	(0,674)	(0,671)	(0,564)	(0,503)	(1,000)	(0,641)	(0,669)	(0,558)	(0,532)

Table .5: 24m periods Average WMSFE comparison of bond prices sorted across maturities groups, when the forecast of existing models is based on a term structure dataset extracted with the Nelson-Siegel-Svennson static model. The percentage comparison respect to the DNS model forecasts is showed in parenthesis.

WRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
03/2004-02/2006															
T 6 5/8 05/15/07	0,234	0,245	0,863**	0,266	0,212	0,301	0,454**	0,864**	0,342**	0,339**	0,389	0,741**	0,391**	0,336	0,316**
	(1,000)	(1,046)	(3,682)	(1,134)	(0,907)	(1,000)	(1,508)	(2,865)	(1,134)	(1,126)	(1,000)	(1,905)	(1,005)	(0,862)	(0,811)
T 6 1/8 08/15/07	0,221	0,244	0,845**	0,233	0,180	0,268	0,427**	0,869**	0,308**	0,302**	0,367	0,728**	0,397**	0,323	0,296**
	(1,000)	(1,100)	(3,817)	(1,053)	(0,814)	(1,000)	(1,591)	(3,238)	(1,149)	(1,126)	(1,000)	(1,984)	(1,081)	(0,880)	(0,806)
T 5 1/2 02/15/08	0,201	0,213	0,819**	0,204	0,155	0,242	0,374**	0,875**	0,270**	0,266**	0,338	0,670**	0,408**	0,296	0,274**
	(1,000)	(1,058)	(4,076)	(1,017)	(0,774)	(1,000)	(1,542)	(3,611)	(1,113)	(1,097)	(1,000)	(1,980)	(1,206)	(0,874)	(0,809)
T 5 5/8 05/15/08	0,191	0,199	0,839**	0,209	0,166	0,245	0,384**	0,905**	0,268**	0,285**	0,323	0,661**	0,426**	0,296**	0,285**
	(1,000)	(1,042)	(4,386)	(1,091)	(0,870)	(1,000)	(1,565)	(3,687)	(1,092)	(1,162)	(1,000)	(2,045)	(1,319)	(0,916)	(0,881)
T 3 1/8 09/15/08	0,179	0,170	0,724**	0,169	0,122	0,236	0,291**	0,792**	0,200**	0,199	0,320	0,549**	0,377	0,244	0,226
	(1,000)	(0,950)	(4,033)	(0,944)	(0,677)	(1,000)	(1,232)	(3,357)	(0,848)	(0,845)	(1,000)	(1,717)	(1,180)	(0,763)	(0,707)
T 4 3/4 11/15/08	0,183	0,185	0,789**	0,188	0,142	0,237	0,355**	0,859**	0,226**	0,243**	0,309	0,607**	0,411	0,269	0,255**
	(1,000)	(1,013)	(4,316)	(1,031)	(0,776)	(1,000)	(1,499)	(3,629)	(0,955)	(1,029)	(1,000)	(1,967)	(1,332)	(0,872)	(0,828)
T 3 3/8 12/15/08	0,186	0,179	0,720**	0,177	0,112	0,232	0,312**	0,787**	0,196**	0,209**	0,305**	0,551	0,378	0,242	0,227
	(1,000)	(0,965)	(3,878)	(0,951)	(0,605)	(1,000)	(1,343)	(3,389)	(0,844)	(0,898)	(1,000)	(1,811)	(1,240)	(0,793)	(0,745)
T 3 1/4 01/15/09	0,193	0,187	0,713	0,189	0,115	0,228	0,305	0,778	0,193	0,201	0,305	0,548	0,374	0,240	0,222
	(1,000)	(0,967)	(3,695)	(0,980)	(0,596)	(1,000)	(1,338)	(3,413)	(0,845)	(0,883)	(1,000)	(1,793)	(1,225)	(0,786)	(0,725)
T 5 1/2 05/15/09	0,196	0,205	0,794**	0,210	0,141	0,253	0,380**	0,850**	0,220**	0,236	0,318	0,614**	0,412	0,267	0,250
	(1,000)	(1,044)	(4,049)	(1,071)	(0,717)	(1,000)	(1,504)	(3,364)	(0,869)	(0,934)	(1,000)	(1,928)	(1,295)	(0,839)	(0,786)
T 6 08/15/09	0,206	0,240**	0,819	0,230	0,135	0,229	0,389**	0,862**	0,238**	0,224**	0,316	0,648**	0,420**	0,280**	0,250**
	(1,000)	(1,165)	(3,971)	(1,115)	(0,656)	(1,000)	(1,698)	(3,762)	(1,038)	(0,977)	(1,000)	(2,054)	(1,332)	(0,889)	(0,791)
T 6 1/2 02/15/10	0,222	0,257**	0,823**	0,250	0,133	0,232	0,404**	0,838**	0,237	0,213**	0,315	0,652**	0,412**	0,276**	0,245**
	(1,000)	(1,157)	(3,712)	(1,128)	(0,599)	(1,000)	(1,747)	(3,622)	(1,023)	(0,921)	(1,000)	(2,070)	(1,308)	(0,877)	(0,776)
T 5 3/4 08/15/10	0,216	0,237**	0,746**	0,265	0,114	0,218	0,359**	0,735**	0,212	0,174**	0,302	0,582**	0,364**	0,236	0,213
	(1,000)	(1,095)	(3,451)	(1,227)	(0,527)	(1,000)	(1,645)	(3,365)	(0,972)	(0,794)	(1,000)	(1,930)	(1,208)	(0,782)	(0,705)

Table .6: 24m periods WMSFE comparison of selected bond prices series, when the forecast of existing models is based on a term structure dataset extracted with the B-Splines static model. The percentage comparison respect to the DNS model forecasts is showed in parenthesis. The ones marked by ** are significantly different with respect to the DNS model under a Diebold-Mariano test with 0.05 significance level.

WRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
12/2012-11/2014															
T 1 7/8 06/30/15	0,146	0,126	0,093**	0,102**	0,101	0,340	0,098	0,101	0,121**	0,075	0,621	0,126**	0,209**	0,140**	0,190**
	(1,000)	(0,862)	(0,635)	(0,699)	(0,697)	(1,000)	(0,289)	(0,298)	(0,356)	(0,220)	(1,000)	(0,203)	(0,337)	(0,225)	(0,306)
T 1 3/8 11/30/15	0,178	0,138**	0,094**	0,120**	0,097	0,399	0,107**	0,110**	0,152**	0,096**	0,704	0,130**	0,264**	0,174**	0,208**
	(1,000)	(0,773)	(0,528)	(0,672)	(0,542)	(1,000)	(0,270)	(0,277)	(0,382)	(0,242)	(1,000)	(0,185)	(0,375)	(0,247)	(0,295)
T 2 1/8 12/31/15	0,173	0,120**	0,077**	0,122**	0,103	0,392	0,098**	0,095**	0,150**	0,089	0,707	0,139**	0,257**	0,173**	0,209**
	(1,000)	(0,694)	(0,443)	(0,707)	(0,592)	(1,000)	(0,250)	(0,242)	(0,383)	(0,226)	(1,000)	(0,197)	(0,364)	(0,245)	(0,296)
T 3 1/4 05/31/16	0,168	0,103**	0,096**	0,135**	0,098	0,406	0,119**	0,107	0,170**	0,103	0,750	0,185**	0,292**	0,198**	0,224**
	(1,000)	(0,615)	(0,572)	(0,802)	(0,584)	(1,000)	(0,292)	(0,264)	(0,419)	(0,253)	(1,000)	(0,247)	(0,389)	(0,265)	(0,299)
T 4 7/8 08/15/16	0,200	0,159	0,155	0,145**	0,118	0,407	0,165	0,155	0,179**	0,108	0,771	0,213**	0,318**	0,211**	0,245**
	(1,000)	(0,793)	(0,776)	(0,726)	(0,590)	(1,000)	(0,406)	(0,381)	(0,439)	(0,266)	(1,000)	(0,276)	(0,412)	(0,273)	(0,317)
T 2 3/4 11/30/16	0,184	0,093**	0,121**	0,153**	0,103**	0,431	0,146**	0,130**	0,206	0,133	0,795	0,237**	0,374**	0,244**	0,255**
	(1,000)	(0,504)	(0,657)	(0,835)	(0,563)	(1,000)	(0,338)	(0,302)	(0,479)	(0,308)	(1,000)	(0,298)	(0,470)	(0,306)	(0,321)
T 4 5/8 02/15/17	0,201	0,160	0,194	0,164**	0,121**	0,424	0,208	0,186	0,216	0,138	0,814	0,286	0,399**	0,257**	0,276**
	(1,000)	(0,794)	(0,965)	(0,814)	(0,603)	(1,000)	(0,490)	(0,438)	(0,509)	(0,324)	(1,000)	(0,351)	(0,490)	(0,316)	(0,340)
T 2 3/4 05/31/17	0,182	0,093**	0,179	0,172**	0,113**	0,439	0,193	0,176	0,246	0,169	0,826	0,314	0,465**	0,291**	0,292**
	(1,000)	(0,509)	(0,984)	(0,946)	(0,622)	(1,000)	(0,441)	(0,402)	(0,560)	(0,385)	(1,000)	(0,380)	(0,563)	(0,353)	(0,354)
T 8 7/8 08/15/17	0,257	0,296	0,386**	0,197**	0,152**	0,445	0,388	0,343	0,268	0,182	0,901	0,483	0,510	0,320	0,342
	(1,000)	(1,151)	(1,497)	(0,767)	(0,591)	(1,000)	(0,870)	(0,771)	(0,602)	(0,409)	(1,000)	(0,537)	(0,567)	(0,355)	(0,379)
T 9 1/8 05/15/18	0,233	0,297**	0,494**	0,223**	0,158**	0,450	0,464	0,412	0,340	0,247	0,957	0,634	0,651	0,399	0,384
	(1,000)	(1,279)	(2,126)	(0,958)	(0,678)	(1,000)	(1,031)	(0,915)	(0,756)	(0,550)	(1,000)	(0,662)	(0,681)	(0,417)	(0,401)
T 4 08/15/18	0,182	0,180	0,368**	0,216**	0,151**	0,425	0,340	0,308	0,340	0,250	0,875	0,526	0,672	0,396	0,389
	(1,000)	(0,989)	(2,017)	(1,183)	(0,829)	(1,000)	(0,800)	(0,724)	(0,801)	(0,590)	(1,000)	(0,601)	(0,768)	(0,452)	(0,445)
T 8 7/8 02/15/19	0,283	0,322	0,596**	0,256**	0,185**	0,461	0,526	0,471	0,405	0,288	0,996	0,747	0,783	0,463	0,466
	(1,000)	(1,135)	(2,105)	(0,904)	(0,652)	(1,000)	(1,140)	(1,020)	(0,879)	(0,625)	(1,000)	(0,750)	(0,786)	(0,465)	(0,468)

Table .7: 24m periods WMSFE comparison of selected bond prices series, when the forecast of existing models is based on a term structure dataset extracted with the B-Splines static model. The percentage comparison respect to the DNS model forecasts is showed in parenthesis. The ones marked by ** are significantly different with respect to the DNS model under a Diebold-Mariano test with 0.05 significance level.

WRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
03/2004-02/2006															
T 6 5/8 05/15/07	0,294	0,275	0,714**	0,266	0,212	0,447	0,416	0,684	0,342**	0,339	0,430	0,461	0,639	0,336	0,316**
	(1,000)	(0,932)	(2,424)	(0,902)	(0,721)	(1,000)	(0,930)	(1,531)	(0,765)	(0,760)	(1,000)	(1,073)	(1,489)	(0,781)	(0,735)
T 6 1/8 08/15/07	0,257	0,239	0,704**	0,233	0,180	0,381	0,333	0,671	0,308	0,302	0,432	0,412	0,630	0,323	0,296
	(1,000)	(0,930)	(2,741)	(0,907)	(0,701)	(1,000)	(0,876)	(1,764)	(0,810)	(0,794)	(1,000)	(0,953)	(1,459)	(0,748)	(0,686)
T 5 1/2 02/15/08	0,234	0,209	0,712**	0,204	0,155	0,365	0,306	0,674	0,270	0,266	0,429	0,397	0,638	0,296	0,274
	(1,000)	(0,893)	(3,048)	(0,874)	(0,665)	(1,000)	(0,838)	(1,848)	(0,739)	(0,729)	(1,000)	(0,925)	(1,487)	(0,690)	(0,639)
T 5 5/8 05/15/08	0,249	0,227	0,755**	0,209	0,166	0,400	0,373	0,714	0,268**	0,285	0,414	0,446	0,679	0,296	0,285
	(1,000)	(0,913)	(3,030)	(0,838)	(0,668)	(1,000)	(0,934)	(1,786)	(0,671)	(0,714)	(1,000)	(1,078)	(1,640)	(0,715)	(0,688)
T 3 1/8 09/15/08	0,216	0,177	0,684**	0,169	0,122	0,349	0,285	0,643	0,200	0,199	0,405	0,376	0,613	0,244	0,226
	(1,000)	(0,819)	(3,171)	(0,785)	(0,563)	(1,000)	(0,816)	(1,844)	(0,574)	(0,572)	(1,000)	(0,930)	(1,515)	(0,603)	(0,558)
T 4 3/4 11/15/08	0,240	0,223	0,769**	0,188	0,142	0,379	0,361	0,724	0,226**	0,243	0,411	0,454	0,692	0,269	0,255
	(1,000)	(0,930)	(3,205)	(0,785)	(0,591)	(1,000)	(0,953)	(1,908)	(0,596)	(0,642)	(1,000)	(1,105)	(1,686)	(0,655)	(0,622)
T 3 3/8 12/15/08	0,239	0,215	0,715**	0,177	0,112	0,357	0,325	0,671	0,196	0,209	0,406	0,425	0,642	0,242	0,227
	(1,000)	(0,900)	(2,995)	(0,739)	(0,470)	(1,000)	(0,911)	(1,878)	(0,549)	(0,584)	(1,000)	(1,048)	(1,581)	(0,595)	(0,559)
T 3 1/4 01/15/09	0,233	0,215	0,721**	0,189	0,115	0,342	0,306	0,676**	0,193	0,201	0,411	0,417	0,648	0,240	0,222
	(1,000)	(0,922)	(3,090)	(0,810)	(0,493)	(1,000)	(0,894)	(1,977)	(0,563)	(0,589)	(1,000)	(1,016)	(1,578)	(0,585)	(0,540)
T 5 1/2 05/15/09	0,258	0,256	0,849**	0,210	0,141	0,387	0,398	0,798**	0,220**	0,236	0,432	0,514	0,769	0,267	0,250
	(1,000)	(0,992)	(3,292)	(0,814)	(0,545)	(1,000)	(1,029)	(2,062)	(0,567)	(0,609)	(1,000)	(1,190)	(1,780)	(0,618)	(0,579)
T 6 08/15/09	0,252	0,271	0,922**	0,230	0,135	0,326	0,368	0,868**	0,238	0,224	0,450	0,531	0,839**	0,280	0,250
	(1,000)	(1,076)	(3,665)	(0,913)	(0,537)	(1,000)	(1,128)	(2,661)	(0,729)	(0,686)	(1,000)	(1,182)	(1,866)	(0,624)	(0,555)
T 6 1/2 02/15/10	0,275	0,307	1,022**	0,250	0,133	0,323	0,410	0,962**	0,237	0,213	0,469	0,595**	0,934**	0,276	0,245
	(1,000)	(1,114)	(3,710)	(0,909)	(0,482)	(1,000)	(1,267)	(2,975)	(0,733)	(0,659)	(1,000)	(1,269)	(1,990)	(0,589)	(0,521)
T 5 3/4 08/15/10	0,279	0,311**	1,039**	0,265	0,114	0,309	0,408	0,977**	0,212	0,174	0,469	0,609**	0,952**	0,236	0,213
	(1,000)	(1,117)	(3,731)	(0,952)	(0,409)	(1,000)	(1,322)	(3,165)	(0,688)	(0,562)	(1,000)	(1,299)	(2,029)	(0,503)	(0,453)

Table .8: 24m periods WMSFE comparison of selected bond prices series, when the forecast of existing models is based on a term structure dataset extracted with the Nelson-Siegel-Svennson static model. The percentage comparison respect to the DNS model forecasts is showed in parenthesis. The ones marked by ** are significantly different with respect to the DNS model under a Diebold-Mariano test with 0.05 significance level.

WRMSFE	h=1					h=6					h=12				
	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS	DNS	FSN-ECM	RW	Bond FSN 2 Ct	Bond-DNSS
12/2012-11/2014															
T 1 7/8 06/30/15	0,272	0,206**	0,133**	0,102	0,101	0,384	0,323	0,211**	0,121**	0,075**	0,580	0,256**	0,485	0,140**	0,190**
	(1,000)	(0,756)	(0,490)	(0,374)	(0,373)	(1,000)	(0,841)	(0,549)	(0,316)	(0,195)	(1,000)	(0,441)	(0,836)	(0,241)	(0,328)
T 1 3/8 11/30/15	0,317	0,246**	0,146**	0,120	0,097	0,431	0,409	0,233**	0,152**	0,096**	0,632	0,268**	0,555	0,174**	0,208**
	(1,000)	(0,778)	(0,461)	(0,377)	(0,305)	(1,000)	(0,947)	(0,541)	(0,353)	(0,223)	(1,000)	(0,425)	(0,879)	(0,275)	(0,329)
T 2 1/8 12/31/15	0,299	0,230	0,126**	0,122**	0,103	0,417	0,417	0,212**	0,150**	0,089**	0,626	0,266**	0,547	0,173**	0,209**
	(1,000)	(0,772)	(0,423)	(0,410)	(0,343)	(1,000)	(0,998)	(0,508)	(0,360)	(0,212)	(1,000)	(0,425)	(0,873)	(0,277)	(0,335)
T 3 1/4 05/31/16	0,277	0,208**	0,111**	0,135**	0,098	0,409	0,467	0,185**	0,170	0,103**	0,639	0,263**	0,560	0,198**	0,224**
	(1,000)	(0,750)	(0,401)	(0,485)	(0,353)	(1,000)	(1,141)	(0,453)	(0,415)	(0,251)	(1,000)	(0,412)	(0,877)	(0,311)	(0,351)
T 4 7/8 08/15/16	0,290	0,236**	0,144**	0,145**	0,118	0,399	0,520	0,188**	0,179	0,108**	0,636	0,276**	0,564	0,211**	0,245**
	(1,000)	(0,815)	(0,497)	(0,501)	(0,407)	(1,000)	(1,305)	(0,472)	(0,448)	(0,271)	(1,000)	(0,434)	(0,887)	(0,331)	(0,385)
T 2 3/4 11/30/16	0,262	0,192**	0,105**	0,153**	0,103	0,407	0,516	0,165**	0,206	0,133**	0,651	0,271**	0,577	0,244**	0,255**
	(1,000)	(0,733)	(0,403)	(0,587)	(0,395)	(1,000)	(1,269)	(0,404)	(0,507)	(0,326)	(1,000)	(0,416)	(0,886)	(0,374)	(0,392)
T 4 5/8 02/15/17	0,264	0,212	0,159**	0,164**	0,121	0,389	0,567	0,171**	0,216	0,138	0,646	0,293	0,572	0,257**	0,276**
	(1,000)	(0,805)	(0,601)	(0,622)	(0,461)	(1,000)	(1,459)	(0,440)	(0,555)	(0,354)	(1,000)	(0,453)	(0,885)	(0,398)	(0,428)
T 2 3/4 05/31/17	0,227	0,156**	0,139**	0,172**	0,113**	0,389	0,546	0,151**	0,246	0,169	0,654	0,292**	0,574	0,291**	0,292**
	(1,000)	(0,690)	(0,615)	(0,759)	(0,499)	(1,000)	(1,404)	(0,389)	(0,632)	(0,435)	(1,000)	(0,447)	(0,877)	(0,445)	(0,447)
T 8 7/8 08/15/17	0,289	0,272	0,335	0,197**	0,152**	0,387	0,690	0,272	0,268	0,182	0,689	0,418	0,586	0,320**	0,342**
	(1,000)	(0,941)	(1,159)	(0,684)	(0,528)	(1,000)	(1,781)	(0,701)	(0,693)	(0,470)	(1,000)	(0,607)	(0,850)	(0,464)	(0,496)
T 9 1/8 05/15/18	0,289	0,261	0,464**	0,223**	0,158**	0,399	0,685	0,362	0,340	0,247	0,750	0,519	0,581**	0,399	0,384**
	(1,000)	(0,903)	(1,608)	(0,772)	(0,546)	(1,000)	(1,717)	(0,906)	(0,852)	(0,619)	(1,000)	(0,692)	(0,774)	(0,532)	(0,512)
T 4 08/15/18	0,207	0,177	0,350**	0,216**	0,151**	0,347	0,611	0,264	0,340	0,250	0,654	0,417	0,534	0,396	0,389**
	(1,000)	(0,857)	(1,692)	(1,044)	(0,732)	(1,000)	(1,762)	(0,761)	(0,981)	(0,722)	(1,000)	(0,639)	(0,817)	(0,605)	(0,595)
T 8 7/8 02/15/19	0,350	0,340	0,607**	0,256**	0,185**	0,414	0,772	0,476	0,405	0,288	0,743	0,619	0,558**	0,463	0,466**
	(1,000)	(0,973)	(1,735)	(0,732)	(0,528)	(1,000)	(1,867)	(1,150)	(0,981)	(0,697)	(1,000)	(0,833)	(0,750)	(0,623)	(0,626)

Table .9: 24m periods WMSFE comparison of selected bond prices series, when the forecast of existing models is based on a term structure dataset extracted with the Nelson-Siegel-Svennson static model. The percentage comparison respect to the DNS model forecasts is showed in parenthesis. The ones marked by ** are significantly different with respect to the DNS model under a Diebold-Mariano test with 0.05 significance level.