Regime Parity

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Abstract

In this paper, we introduce a new risk-based, long-run portfolio that bridges the gap between the multi-regime distribution of assets and risk-parity investing. Building on the seminal idea of the risk-parity portfolio, we relate regime-level information and portfolio allocation. When returns are influenced by latent regimes such as recession-expansion phases or inflation shocks, these non-stationarities can affect the unconditional distribution underlying the portfolio construction, leading the risk-parity allocation to load too much risk on the hedging asset. Our solution is the regime-parity portfolio, which is a linear combination between regime-specific risk-parity portfolios. We propose to weigh these regime-specific portfolios according to the steady-state probabilities of each regime, thus making use of the entire regime-level information. This new portfolio shows interesting features, notably better resistance to rare, yet adverse, regimes. We present the model and its salient features before showcasing different practical applications, highlighting the empirical interest of the approach.

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Section 1 - Introduction

Multi-asset portfolio construction is often split into two components: a long-run portfolio and a tactical overlay. The long-run (potentially long-only) portfolio captures the unconditional multivariate distribution of financial assets, while the tactical overlay tilts the long-run portfolio from its long-term equilibrium to adapt the final portfolio to the current regime guiding markets.

A well-established empirical fact in financial returns is the existence of low frequency alternating regimes making the dynamic of financial assets non-linear. For example, Maheu and McCurdy (2000) study the properties of bull and bear markets and Guidolin and Timmermann (2006) consider a three-regime framework when modelling the joint dynamics of stocks and bonds. While timing these regimes is at the heart of constructing the tactical overlay – as highlighted in Van Vliet and Blitz (2011) and Ielpo (2014) for traditional risk premia or Blin et al. (2020) for alternative risk premia – the long-run portfolio construction usually omits these regimes. Instead, it is designed to obtain an unconditionally optimal portfolio derived from the unconditional covariance-matrix.

Among the long-run portfolio candidates, risk parity has gained in popularity in the recent years to become a standard in the financial industry. The decades-long decline in yields since the 1990s has aided the credibility of fixed income for diversification purposes.

The idea of this portfolio construction is to allocate an equal weight between assets, in term of risk rather than capital, to improve diversification. However, as emphasised by Ardia et al. (2018), the resulting portfolio, although diversified in risk, tends to be concentrated in term of its contribution to the final performance-to-risk ratio (for example, the Sharpe ratio). In other words, while empirically well distributed in terms of risk, such a portfolio will become progressively over-reliant on its diversifying assets as sources of returns.

In this paper, we introduce a new risk-based, long-run portfolio that bridges the gap between the multi-regime distribution of assets and risk-parity investing. The first section explicitly derives this regime-parity portfolio, building upon the standard Equal Risk Contribution approach. In section 2, we illustrate the sensitivity of our allocation to different shocks; in particular we show that our portfolio construction is more resilient to structural breaks and infrequent tail-risk events. Section 3 applies our findings to different markets. The final section concludes and surveys potential future research perspectives.

Section 2 - Risk-based portfolio construction with regime switching dynamics

Long-run portfolios are designed to harvest the long-term performance of assets. For that reason, they are often computed from the unconditional distribution of returns, enabling stable allocation while active bets are effectively concentrated in the tactical overlays. As this long-run allocation is the foundation of the portfolio, it should be diversified enough to efficiently harvest a wide range of risk premia.

Capital allocation is known to be deceiving from that perspective: what looks well diversified in terms of capital allocation can present a poor breakdown of risk across risk premia. For example, a standard 60/40 US equity-bond portfolio often hides highly concentrated risk in equities as the asset class accounts for more than 90% of the total portfolio risk.

Allocation risk, rather than capital, is a preferable solution that constructs portfolios by considering the risk contribution of each risk premia to the total portfolio risk and explicitly helps improve diversification. Cases in point are the famous risk parity or Equal Risk Contribution (ERC) portfolios introduced by Maillard et al. (2010) and Bai et al. (2016). The latter consider volatility as the risk measure while Boudt et al. (2012) generalise the risk parity to downside risk measures such as the Value-at-Risk.

The derivation of the ERC portfolio is based on the decomposition of the total risk of the portfolio, denoted $\mathcal{R}_P(\boldsymbol{\omega})$ where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N)$ denotes the capital allocation, into the individual risk contributions[2] :

$$\mathcal{R}_P(\boldsymbol{\omega}) \; = \sum_{i=1}^{N} \omega_i \frac{\partial \mathcal{R}_P(\boldsymbol{\omega})}{\partial \omega_i} = \sum_{i=1}^{N} C_i^{\mathcal{R}}(\boldsymbol{\omega}).$$

The ERC portfolio is thus the portfolio $\boldsymbol{\omega}^*$such that $\sum_{i=1}^{N} \omega_i^* = 1$ and $C_1^{\mathcal{R}}(\boldsymbol{\omega}^*) = \cdots = C_N^{\mathcal{R}}(\boldsymbol{\omega}^*)$. Computing individual risk contributions, which are functions of the unconditional moments of the multivariate distribution of asset returns, helps derive weights, making risk contributions equal.

However, when the asset dynamics are regime-dependent, these unconditional moments have a tendency to crush this regime-level information, when it should inform us about the long-term behaviour of asset returns. To illustrate this inconsistency of unconditional moments, there is no better example than a mixture of Gaussian distributions.

Figure 1 presents two of such examples where a random variable X follows a Gaussian mixture with equal weights between regime 1 where $X \sim \mathrm{N}(\mu_1, \sigma_1)$ and regime 2 where $X \sim \mathrm{N}(\mu_2, \sigma_2)$. Depending on the parameters of the distribution, note how the Gaussian distribution implied by the unconditional moments of X can be uninformative, failing to reveal information about the

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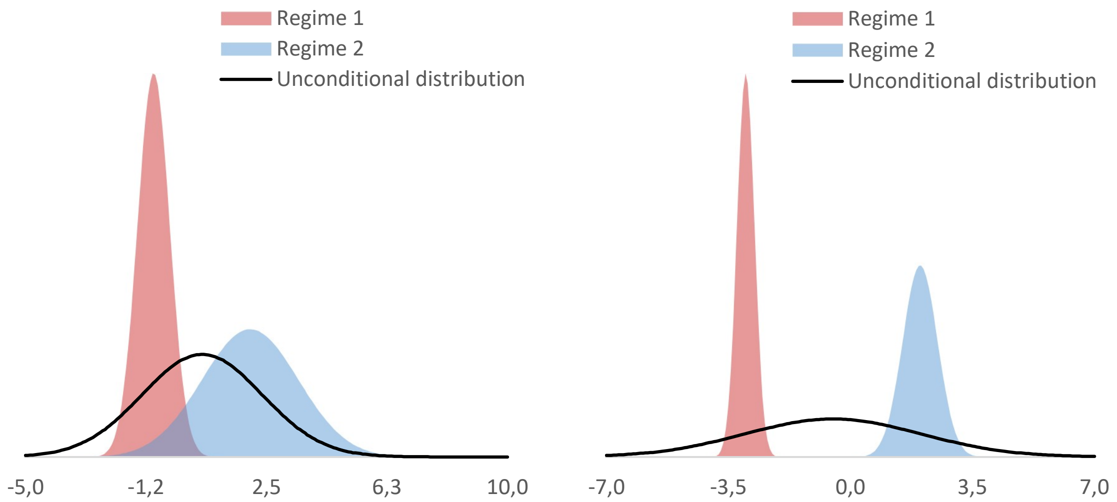
[2] From Euler homogeneous function theorem, if a function $f: \mathbb{R}^N \to \mathbb{R}$ is first-degree homogenous (ie verifying $f(ax) \; = \; a\, f(x)$), then we have the following decomposition: f(x) = $\sum_{i=1}^{N} x_i\, \partial f(x)/\partial x_i$.

true distribution. For example, when $\mu_1 = -3$, $\sigma_1 = 0.25$, $\mu_2 = 2$ and $\sigma_2 = 0.5$, the implied unconditional distribution indicates a probability of 30% for X to be between -2 and 0 although the actual distribution of X has a zero probability on this interval.

Figure 1. Example of mixture distributions with their implied Gaussian unconditional distributions



$\mu_1 = -1,\ \sigma_1 = 0.5,\ \mu_2 = 2$ and $\sigma_2 = 1.5$ $\quad$ $\mu_1 = -3,\ \sigma_1 = 0.25,\ \mu_2 = 2$ and $\sigma_2 = 0.5$

Source: LOIM. For illustrative purposes only.

To mitigate the effect of regimes on a long-run risk-based portfolio, we introduce a new portfolio that accounts for the non-stationarities of the returns, induced by changing regimes, explicitly using this regime-level information to build a regime-resistant long-run allocation. Our regime-parity portfolio preserves the risk-diversification features of the risk-parity portfolio by relying on a regime-specific ERC solution but renders explicit the embedding of regimes by weighting the local ERC portfolios by the probability of each regime occurring. Thus, the regime-parity portfolio is a linear combination given by

$$\boldsymbol{\varpi}^* = \sum_{j=1}^{M} \pi_j \boldsymbol{\omega}_j^*$$

where M denotes the number of regimes, π_j the steady-state probability of regime $j = 1, \dots, M$, and $\boldsymbol{\omega}_j^*$ is the ERC portfolio computed on the multivariate distribution conditional on being in regime j. Of course, if the regimes are uninformative about the distribution of returns, the regime-parity portfolio coincides with the ERC portfolio as the conditional distributions are identical to the unconditional distribution. When such regimes exist and discriminate with respect to the data-generating process that animates returns, however, the two portfolios deviate from each another.

The probability-weighting of the local ERC portfolios has the advantage of being linear in the frequency of occurrence. A similar weighting scheme was used by Maheu and McCurdy (2009) when modelling the long-run distribution of equity returns in the presence of structural breaks. Interestingly, it also relates to the probability matching effect in the behavioural finance literature (see, for example, Kim and Kim (2022)) that influenced the adaptive markets hypothesis of Lo (2004).

Section 3 - Sensitivities of the regime-parity portfolio

To illustrate the differences between the regime-parity and the risk-parity portfolios, we first consider a synthetically simplified example, considering volatility as the risk measure to break down. We assume that our investment universe is composed of a risky asset and a hedging asset that follow a conditionally Gaussian multivariate regime-switching model with two regimes: a stressed regime that rarely occurs and a calm regime that occurs frequently. More precisely, we assume the following dynamic for the assets' excess returns:

$$\boldsymbol{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \end{pmatrix} \sim \begin{cases} \mathrm{N}\left(\boldsymbol{\mu_1}, \begin{pmatrix} \sigma_{1,1} & 0 \\ 0 & \sigma_{1,2} \end{pmatrix} \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,1} & 0 \\ 0 & \sigma_{1,2} \end{pmatrix}\right) & if\ s_t = 1 \\ \mathrm{N}\left(\boldsymbol{\mu_2}, \begin{pmatrix} \sigma_{2,1} & 0 \\ 0 & \sigma_{2,2} \end{pmatrix} \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{2,1} & 0 \\ 0 & \sigma_{2,2} \end{pmatrix}\right) & if\ s_t = 2 \end{cases}$$

where

$$\begin{pmatrix} r_{1,t} \\ r_{2,t} \end{pmatrix} \sim \begin{cases} \mathrm{N}\left(\begin{matrix} 2\% \\ 10\% \end{matrix}, \begin{pmatrix} 3\% & 0 \\ 0 & 15\% \end{pmatrix} \begin{pmatrix} 1 & -0.2 \\ -0.2 & 1 \end{pmatrix} \begin{pmatrix} 3\% & 0 \\ 0 & 15\% \end{pmatrix}\right) & if\ s_t = 1 \\ \mathrm{N}\left(\begin{matrix} 5\% \\ -35\% \end{matrix}, \begin{pmatrix} 5\% & 0 \\ 0 & 35\% \end{pmatrix} \begin{pmatrix} 1 & -0.4 \\ -0.4 & 1 \end{pmatrix} \begin{pmatrix} 5\% & 0 \\ 0 & 35\% \end{pmatrix}\right) & if\ s_t = 2 \end{cases}$$

and s_t is a Markov chain with the following transition probabilities matrix

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{pmatrix}$$

yielding the steady state probabilities $\pi_1 = 0.91$ and $\pi_2 = 0.09$. In the case of two assets, the solution of the ERC is explicit and given by

$$\omega_1^* = \frac{\sigma_2}{\sigma_1 + \sigma_2}\ and\ \omega_2^* = \frac{\sigma_1}{\sigma_1 + \sigma_2}.$$

Additionally, the unconditional mean of $\boldsymbol{r}_t$ is given by $\boldsymbol{\mu_0} = \pi_1 \boldsymbol{\mu_1} + \pi_2 \boldsymbol{\mu_2}$ while the unconditional individual volatility of each asset $j = 1,2$ is given by

$$\sigma_{0,j} = \sqrt{\pi_1(\sigma_{\boldsymbol{1},\boldsymbol{j}}^2 + \mu_{\boldsymbol{1},\boldsymbol{j}}^2) + \pi_2(\sigma_{\boldsymbol{2},\boldsymbol{j}}^2 + \mu_{\boldsymbol{2},\boldsymbol{j}}^2) - \mu_{\boldsymbol{0},\boldsymbol{j}}^2}.$$

We can thus derive analytically the unconditional ERC portfolio and the regime-parity portfolio. Table 1 shows the weight of each asset in the different portfolios. Note that since the Sharpe ratio of the two assets are equal in regime 1 (the calm regime), the regime-specific ERC is locally optimal in the sense of Markowitz (see Maillard et al. (2010) and Ardia and Boudt (2015)). In that sense, the deviation of the unconditional ERC portfolio from the portfolio in regime 1 appears disproportionate with regards to the low probability of the stress regime occurring. On the contrary, the regime-parity portfolio deviates less while still adding some hedging assets to the portfolio to account for the possible occurrence of the stressed regime.

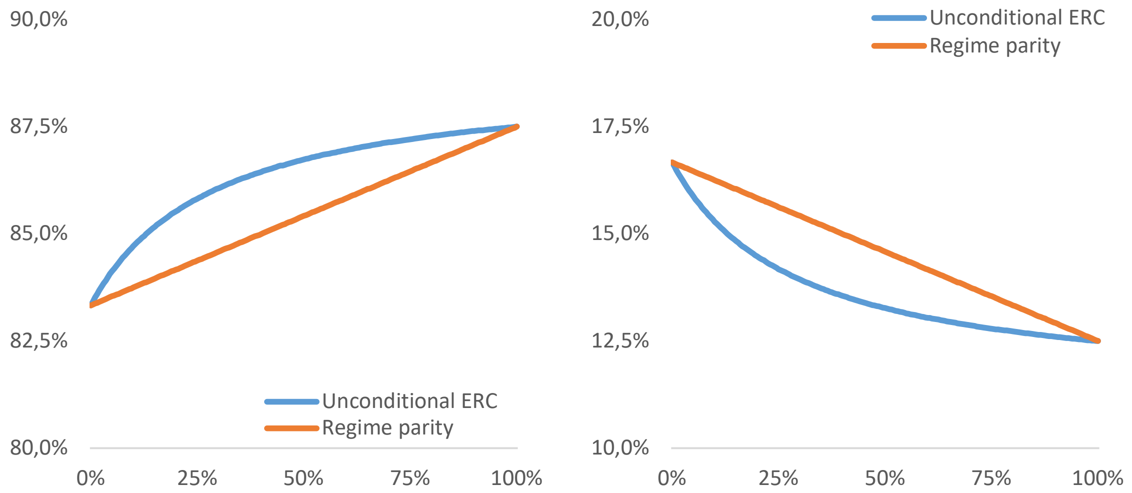
Table 1. Weights of the different portfolios

	ERC regime 1	ERC regime 2	ERC unconditional	Regime parity
Asset 1	83.33%	87.50%	86.79%	83.71%
Asset 2	16.67%	12.50%	13.21%	16.29%

Source: LOIM. For illustrative purposes only.

To analyse this deviation, figure 2 presents the weights of the hedging asset and the risk asset as a function of the steady-state probability of occurrence of the stress regime π_2. Particularly striking is the very large deviation of the unconditional risk-parity portfolio, even for highly infrequent stressed regimes. On the contrary, the regime parity is by construction linearly updating with the unconditional regime probability and, linearly, goes from the risk-parity portfolio in regime 1 to the risk-parity solution in regime 2 as the regime becomes more frequent. This non-linear effect of the risk-parity portfolio stems from the inability of the unconditional moments to correctly capture both modes of the mixture distribution, whereas the regime-parity construction alleviates this effect.

Figure 2. Portfolio composition as a function of the frequency of occurrence of the stress regime



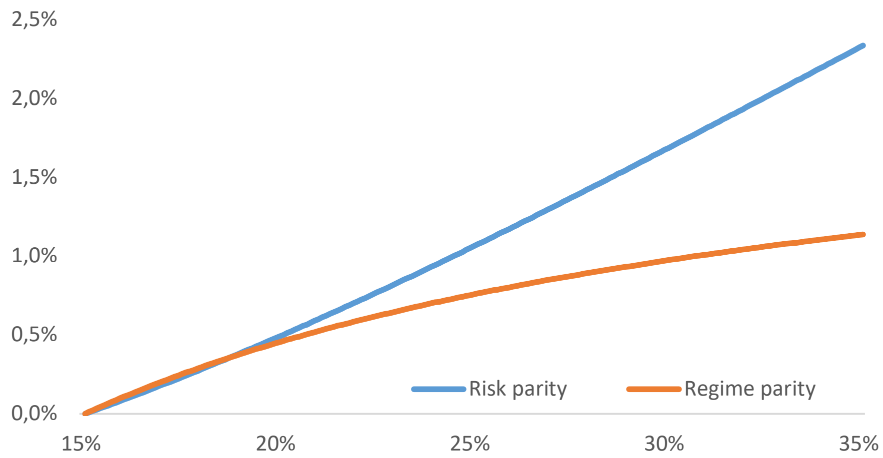
Weight of the hedging asset **Weight of the risk asset**

Source: LOIM. For illustrative purposes only.

While the linearity-in-probability feature of the regime parity is helpful when tracking the individual regime-specific risk contributions in the total risk of the portfolio, the non-linearity of the unconditional ERC may prove useful when large shocks occur in the stress regime.

Figure 3 provides a counterargument, however. While the unconditional ERC presents a concave response to the increase in the stressed regime probability, its response to an increase in the magnitude of the risk in the stress regime is linear. Indeed, the figure presents the relative increase in the allocation to the hedging asset as a function of the risk asset volatility in the stress regime, keeping the frequency occurrence of the stress regime constant at 9%. In this case, the regime-parity portfolio presents a desired concave profile, being able to respond quickly to an increase in risk in the adverse regime but without distorting too sharply the final portfolio from the optimal portfolio in the more frequent calm regime. On the contrary, the unconditional ERC portfolio presents a linear sensibility to the risk in the adverse regime, artificially augmenting the weight of the hedging asset.

Figure 3. Portfolio allocation to the hedging asset as a function of the risk asset volatility in an adverse regime

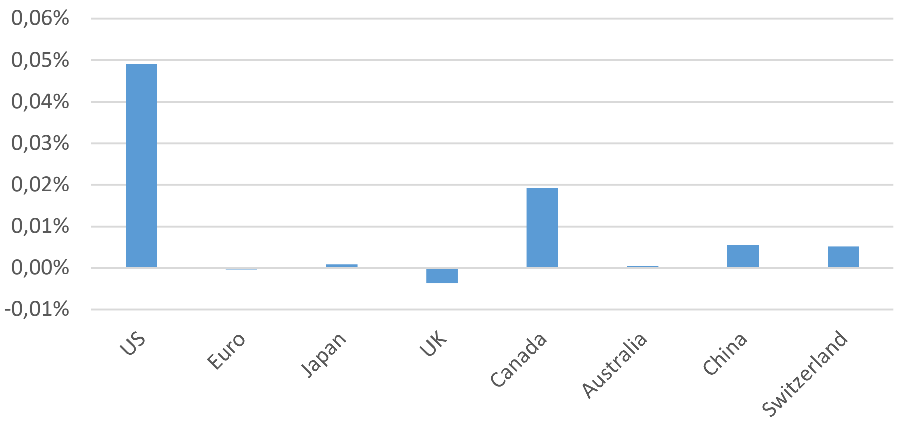


Source: LOIM. For illustrative purposes only.

Section 4 - Empirical applications

To illustrate the advantages of our portfolio construction, we present a set of applications based on real data. We consider different geographical markets (US, Eurozone, Japan, UK, Canada, Australia, China and Switzerland), each with three assets: a local equity index, a local sovereign bond index, and a local investment-grade corporate debt index. We assume a two-regime economy with expansion and recession phases. For the regime classification, we follow the NBER (National Bureau of Economic Research) and the CEPR (Centre for Economic Policy Research) for the US and the Eurozone respectively.We derive the regimes in the other economies through a standard regime-switching method applied to a set of classic business cycle domestic measures as there exists no consensus for dating recessions. Figure 4 presents the annualised outperformance of the regime-parity portfolio compared to the unconditional risk-parity portfolio.

Figure 4. Annualised outperformance of the regime-parity portfolio against the unconditional ERC portfolio

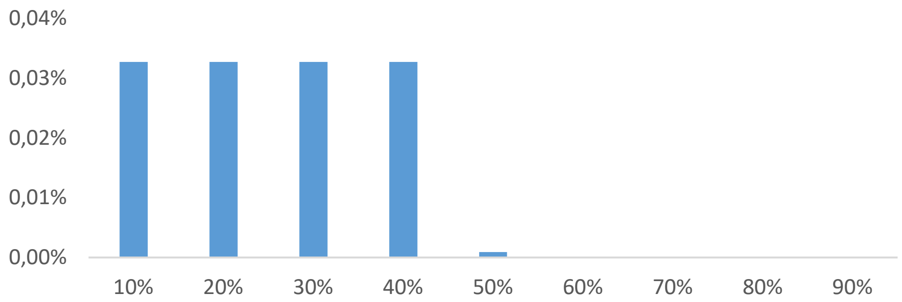


Source: LOIM. For illustrative purposes only.

Although they prove interesting in the US and Canada, the results appear relatively disappointing for the other markets. We postulate that this absence of outperformance stems from the misidentification of regimes. Indeed, if regimes are uninformative for the conditional distribution of returns, the regime-parity portfolio coincides with the ERC portfolio, yielding the same performance in the end.

To illustrate the sensitivity of the regime-parity portfolio construction to the identification of regimes, we carry out a set of Monte Carlo experiments. Building on the illustrative model presented in section 2, we randomly and purposefully mis-specify a proportion λ of the regime identification, meaning that a proportion λ of the points used to compute the conditional moments are not correctly classified. Figure 5 presents the outperformance of the regime-parity portfolio compared to the unconditional risk-parity portfolio as a function of λ. We see that although the regime-parity portfolio is relatively resilient to mis-specified regimes, the outperformance disappears when the false discovery rate in the identification of the regime becomes large.

Figure 5. Outperformance of the regime-parity portfolio against the unconditional ERC portfolio as a function of regime mis-specification



Source: LOIM. For illustrative purposes only.

Section 5 - Conclusion: improving the return-contribution balance

The risk-parity approach provides a coherent allocation scheme to create diversified long-run portfolios. However, when returns are influenced by latent regimes such as recession-expansion phases or inflation shocks, these non-stationarities can affect the unconditional distribution underlying the portfolio construction, leading the risk-parity allocation to load too much risk on the hedging asset.

Our solution is the regime-parity portfolio, which is a linear combination between regime-specific risk-parity portfolios. We propose to weigh these regime-specific portfolios according to the steady-state probabilities of each regime, thus making use of the entire regime-level information. This new portfolio shows interesting features, notably better resistance to rare, yet adverse, regimes. The success of such an investment approach depends significantly on the accuracy of the regime retained to split the history of returns. This portfolio should clearly appeal to multi-asset portfolio managers because it improves the return-contribution balance within their long-run allocations.

Section 7 – References

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