What can you really tell from option prices?

Oleg Bondarenko, Yannick Dillschneider, Paul Schneider, and Fabio Trojani

January 26, 2024

Abstract

Recent theoretical and empirical work exploits option-implied information to constrain the conditional moments of asset returns. In this paper, we characterize the information content that can be reliably used for this purpose under an option market's incompleteness. Our analysis relies on novel upper and lower bounds for a broad family of risk-neutral moments built from a given cross-section of option prices. We show the non-robustness of several risk-neutral moments in the literature, form new robust versions of them, and derive corresponding model-free bounds on the conditional moments of asset returns. These new bounds deviate substantially, both quantitatively and qualitatively, from those associated with non-robust risk-neutral moments.

1 Introduction

Ever since the seminal Breeden and Litzenberger (1978) result, relating derivatives of option prices to the risk-neutral density of the underlying, a rapidly expanding literature has motivated and tested empirically several bounds on conditional physical moments of returns resulting from risk-neutral moments. Given the increasing liquidity and availability of options across many different markets, these assumptions yield potentially powerful economic information about the *conditional moments of asset returns and the unobserved stochastic discount factor*. These statements typically rely on the assumption of a complete market spanned continuously by option portfolios. Such an assumption is usually not met in empirical applications, as observed option cross sections are discrete and finite. As a consequence, infinitely many risk-neutral probability distributions are compatible with observed option cross sections. The implications of this market incompleteness, and its consequences for empirical work with option-implied moments, are the subject of this paper.

Our analysis relies on suitable optimization problems that characterize the maximal and minimal price of a target payoff, across all risk-neutral probability measures that price a given option cross section. It turns out, that neither the maximal nor the minimal such price needs to be finite, and their difference is a natural indicator for the robustness of the price of the option replication portfolio for the given payoff. Two prominent non-robust examples arise for the VIX^2 payoff from CBOE (2009), and the SVIX^2 payoff introduced in Carr and Corso (2001) and later Martin (2017).

Using our upper and lower bounds, we empirically find that the incompleteness arising from the discreteness of strikes is negligible for the densely quoted S&P 500 options. However, the incompleteness induced by lacking option prices outside of the observed strike region is sizable enough, to substantially

Addresses: O. Bondarenko, University of Illinois at Chicago, olegb@uic.edu; Y. Dillschneider, University of Amsterdam, y.dillschneider@uva.nl; P. Schneider, USI Lugano and Swiss Finance Institute, paul.schneider@usi.ch; F. Trojani, University of Geneva and Swiss Finance Institute, fabio.trojani@unige.ch.

affect magnitude and variation of both computed implied moments and the resulting constraints for the conditional moments of asset returns.

Methodologically, our theoretical results extend previous work in Lee (2004), Hobson and Klimmek (2012), and Davis et al. (2014) on options at extreme strikes, as well as results on the sub- and super replication of trading strategies in presence of bid-ask spreads. We demonstrate that they have important implications for empirical work based on risk-neutral expectations computed from option prices, such as Schneider 2015, Martin (2017), Martin and Wagner (2019), Schneider and Trojani (2019), and Chabi-Yo et al. (2023), among many others.

2 Introductory examples

Consider a tradeable asset whose (forward) gross return over some time horizon τ is denoted by R and takes values in $\mathcal{X} = (0, \infty)$. The economy features an arbitrage-free option market, where plain-vanilla European put and call options on R with time-to-maturity τ and strikes $k \in \mathbb{K} = [\ell, u]$ for $0 < \ell \leq 1 \leq u < \infty$ are traded. Observed (forward) put and call prices are denoted by $P(k)$ and $C(k)$, respectively, assuming that $P(\ell) > 0$ and $C(u) > 0$.

Given the absence of arbitrage opportunities within an incomplete market setting, there exists a nontrivial set $\mathcal{B}$ of risk-neutral probability measures that are consistent with observed put and call prices. Specifically, for any admissible $\mathbb{Q} \in \mathcal{B}$, the put prices $P^{\mathbb{Q}}(k)$ and call prices $C^{\mathbb{Q}}(k)$ associated to $\mathbb{Q}$ satisfy $P^{\mathbb{Q}}(k) = P(k)$ and $C^{\mathbb{Q}}(k) = C(k)$ for all $\ell \leq k \leq u$, while the tail behavior of $P^{\mathbb{Q}}(k)$ and $C^{\mathbb{Q}}(k)$ at strikes $0 < k < \ell$ and $k > u$ is not uniquely determined.

By well-established spanning properties, options are particularly informative about the underlying market, so that prices of nonlinear payoffs can be represented as prices of option portfolios (see Bakshi et al. 2003; Bick 1982; Britten-Jones and Neuberger 2000; Carr and Madan 2001). Concretely, for a (almost-everywhere) twice continuously differentiable function f and any admissible risk-neutral probability measure $\mathbb{Q} \in \mathcal{B}$, it follows:

$$\begin{aligned} \mathrm{E}^{\mathbb{Q}}[f(R)] &= f(1) + \int_0^1 f''(K)\, P^{\mathbb{Q}}(K)\, \mathrm{d}K + \int_1^\infty f''(K)\, C^{\mathbb{Q}}(K)\, \mathrm{d}K \\ &= f(1) + \int_\ell^1 f''(K)\, P(K)\, \mathrm{d}K + \int_1^u f''(K)\, C(K)\, \mathrm{d}K \\ &\quad + \underbrace{\int_0^\ell f''(K)\, P^{\mathbb{Q}}(K)\, \mathrm{d}K}_{\mathcal{P}^{\mathbb{Q}}[f]} + \underbrace{\int_u^\infty f''(K)\, C^{\mathbb{Q}}(K)\, \mathrm{d}K}_{\mathcal{C}^{\mathbb{Q}}[f]}\,. \end{aligned} \tag{2.1}$$

Except for the tail integrals $\mathcal{P}^{\mathbb{Q}}[f]$ and $\mathcal{C}^{\mathbb{Q}}[f]$, all terms in this identity are uniquely identified from the observed option prices. Hence, the range of possible values for the associated option-implied moment (or arbitrage-free valuation) $\mathrm{E}^{\mathbb{Q}}[f(R)]$ crucially depends on the tail behavior of $P^{\mathbb{Q}}(k)$ and $C^{\mathbb{Q}}(k)$ under admissible probabilities $\mathbb{Q} \in \mathcal{B}$. The key question studied in this paper concerns the properties of payoffs $f(R)$ that yield a bounded or an unbounded range of admissible valuations.

The mechanism underlying our analysis is easily understood from the static replication formula (2.1). To illustrate it in a simplified setting, we explicitly construct and investigate a parametric subfamily of probability measures $\mathbb{Q}_{a,b} \in \mathcal{B}$, which depend on parameters a, b controlling the left- and right-tail behavior of option prices. Each probability $\mathbb{Q}_{a,b}$ is constructed to ensure that for sufficiently small $0 < a \leq \ell$ and large $b \geq u$ it yields admissible arbitrage-free extrapolations of tail option prices that are piecewise-linear:

$$P^{\mathbb{Q}_{a,b}}(k) = \frac{(k-a)^+}{\ell - a} P(\ell) \ , \qquad C^{\mathbb{Q}_{a,b}}(k) = \frac{(b-k)^+}{b-u} C(u) \ . \tag{2.2}$$

Details on the construction of probabilities $\mathbb{Q}_{a,b}$ are provided in appendix A.1, while additional results for option prices with power-law tails, which lead to fully analogous conclusions, are collected in appendix A.2.

Given tail behavior equation (2.2), tail integrals $\mathcal{P}^{\mathbb{Q}_{a,b}}[f]$ and $\mathcal{C}^{\mathbb{Q}_{a,b}}[f]$ are given in closed-form, after performing integration by parts:

$$\mathcal{P}^{\mathbb{Q}_{a,b}}[f] = -P(\ell)\left(\frac{f(\ell)-f(a)}{\ell-a} - f'(\ell)\right) \tag{2.3}$$

$$\mathcal{C}^{\mathbb{Q}_{a,b}}[f] = C(u)\left(\frac{f(b)-f(u)}{b-u} - f'(u)\right) \ . \tag{2.4}$$

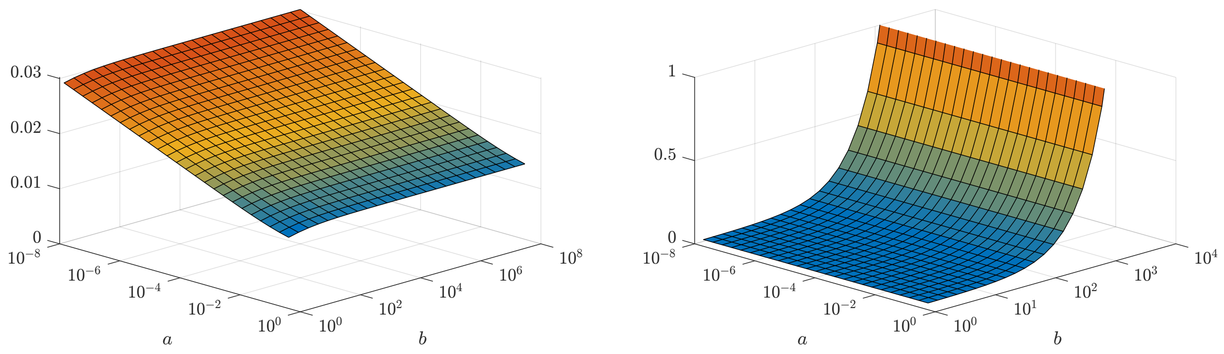
The benchmark case where a, b tend towards their corresponding state space boundaries is particularly relevant. To investigate it, we suppose for simplicity of the argument that payoff function f is convex or concave in each tail, and get:

$$\lim_{a\to 0} \mathcal{P}^{\mathbb{Q}_{a,b}}[f] = -P(\ell)\left(\frac{f(\ell) - \lim_{a\to 0} f(a)}{\ell} - f'(\ell)\right) \tag{2.5}$$

$$\lim_{b\to\infty} \mathcal{C}^{\mathbb{Q}_{a,b}}[f] = C(u)\left(\lim_{b\to\infty} \frac{f(b)}{b} - f'(u)\right) \ . \tag{2.6}$$

Whenever $P(\ell), C(u) > 0$, these limits are completely determined from the tail behavior of f. Equation (2.5) shows that left-tail integral $\mathcal{P}^{\mathbb{Q}_{a,b}}[f]$ is finite if and onyl if f is bounded for small return realizations: $\lim_{a\to 0} |f(a)| < \infty$. Analogously, from equation (2.6) right-tail integral $\mathcal{C}^{\mathbb{Q}_{a,b}}[f]$ is finite if and only if f grows at most linearly for large return realizations: $\lim_{b\to\infty} \frac{|f(b)|}{b} < \infty$.

Naturally, each parameter choice a, b produces a different value of implied valuation $\mathrm{E}^{\mathbb{Q}_{a,b}}[f(R)]$ in equation (2.1). Therefore, $\mathrm{E}^{\mathbb{Q}_{a,b}}[f(R)]$ is bounded across feasible parameter choices a, b if and only if both $\mathcal{P}^{\mathbb{Q}_{a,b}}[f]$ and $\mathcal{C}^{\mathbb{Q}_{a,b}}[f]$ are bounded. However, it is stunningly simple to come up with payoffs $f(R)$ for which at least one tail integral is not finite, and most payoffs commonly employed in the literature actually have this property. One such example is given by VIX and SVIX payoffs, for which we provide some first benchmark empirical evidence in figure 2.1, using historical averages of S&P 500 option prices (cf. appendix C).



(a) VIX **(b)** SVIX

Figure 2.1: Tail behavior of VIX and SVIX

This figure reports option-implied moments $\mathrm{E}^{\mathbb{Q}_{a,b}}[f(R)]$ for VIX and SVIX payoffs and different parameter choices a, b. For the VIX, panel (a) employs payoff $f(R) = \mathrm{VIX}^2(R)$ as in example 2.1. For the SVIX, panel (b) employs payoff $f(R) = \mathrm{SVIX}^2(R)$ as in example 2.2. Option-implied moments are computed using average S&P 500 option prices with quarterly maturity.

Example 2.1 (VIX)**.** To define the VIX as in CBOE (2009), we use its relation to the log contract (e.g., Carr and Wu 2009). Concretely, we set the convex payoff $\mathrm{VIX}^2(R) = -2\log R$ and note that $\lim_{a\to 0} -2\log a = \infty$ and $\lim_{b\to\infty} \frac{-2\log b}{b} = 0$. Accordingly, the left-tail integral $\mathcal{P}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2]$ is positive and increasing as $a \to 0$ with infinite limit in equation (2.5),

$$\lim_{a\to 0} \mathcal{P}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2] = 2P(\ell)\left(\frac{\log \ell - \lim_{a\to 0}\log a}{\ell} - \frac{1}{\ell}\right) = \infty \, .$$

In contrast, the right-tail integral $\mathcal{C}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2]$ is positive and increasing as $b \to \infty$, but with finite limit in equation (2.6) because of the sublinear growth in the right tail. Hence, due to its unboundedness in the left tail, there is no finite upper bound for the VIX valuations $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2(R)]$. Panel (a) in figure 2.1 shows the quantitative behavior of $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2(R)]$ for various combinations of a and b using average S&P 500 option prices. Consistent with the properties of the tail integrals, changes in b have comparatively little effect on the valuation, whereas it increases linearly (on log scale) as $a \to 0$. Even though the dependence on a may appear rather moderate, it should be noted that valuations more than double from the largest to the smallest a reported. ■

Example 2.2 (SVIX)**.** For the SVIX^2 in Carr and Corso (2001) and Martin (2017), we define the convex payoff $\mathrm{SVIX}^2(R) = (R-1)^2$ and note that $\lim_{a\to 0}(a-1)^2 = 1$ and $\lim_{b\to\infty} \frac{(b-1)^2}{b} = \infty$. Hence, the left-tail integral $\mathcal{P}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2]$ is positive and increasing as $a \to 0$ with finite limit in equation (2.5), as the payoff is bounded in the left tail. However, the right-tail integral $\mathcal{C}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2]$ is positive and increasing as $b \to \infty$ with infinite limit in equation (2.6),

$$\lim_{b\to\infty} \mathcal{C}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2] = C(u)\left(\lim_{b\to\infty}\frac{(b-1)^2}{b} - 2(u-1)\right) = \infty \, .$$

Hence, due to its superlinear growth in the right tail, there is no finite upper bound for the SVIX valuations $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2(R)]$. Panel (b) in figure 2.1 reports numerical values of $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2(R)]$ for a range of combinations a and b, again computed from average S&P 500 option prices. As expected from our analysis of the tail integrals, now changes in a have only a minor effect on the valuation, while letting

$b \to \infty$ leads to an exponential increase (on log scale). Hence, the sensitivity of SVIX valuations to option price tails is much larger compared to that of VIX valuations. ■

The above derivations in a simplified context underscore the fact that in an incomplete option market, a multitude of risk-neutral probability measures exists that are all compatible with observed option prices. Without further assumptions, the valuation of a newly issued derivative payoff is not unique. In such settings, robust payoffs are those that imply a bounded range of valuations under all admissible risk-neutral distributions. Non-robust payoffs are those that imply an unbounded range of valuations.

Armed with the necessary intuition, we derive in the sequel results for general arbitrage-free economies under minimal assumptions on the given market structure.

3 Robust option-implied moments

In this section, we present a general approach that yields model-free bounds on option-implied moments, or arbitrage-free valuations, relying on only minimal structure. Under simplifying assumptions regarding the option market, the lower and upper bounds are available in closed form, depending only on the local behavior of option prices at extreme strikes and the tail behavior of the payoff in question. Based on the structure of these bounds, we then provide a characterization of robust payoffs. Investigating wide classes of payoffs that are commonly employed in the literature, we show that most of them actually fail to be robust.

3.1 Model-free bounds on option-implied moments

As in section 2, we denote by R the τ-period forward gross return of a tradeable asset from time t to $T = t + \tau \geq t$. As both t and T are fixed, we omit them below to lighten notation. Without imposing any prior restrictions on possible return realizations, we suppose that the gross return R takes values in $\mathcal{X} = (0, \infty)$.[1] Note that working with forward gross returns naturally encompasses also economies with stochastic interest rates. We associate with the gross return the set of risk-neutral probability measures

$$\mathcal{A} := \left\{ \text{probability measure } \mathbb{Q} \text{ on } \mathcal{X} : \mathrm{E}^{\mathbb{Q}}[R] = 1 \right\} ,$$

where $\mathrm{E}^{\mathbb{Q}}[\cdot]$ denotes the time-t conditional expectation under the measure $\mathbb{Q}$. The set $\mathcal{A}$ reflects all no-arbitrage pricing constraints implied by the underlying market.

The underlying market is accompanied by an arbitrage-free option market, in which we observe the time-t forward prices $P(k)$ and $C(k)$ of traded European put and call options on R with maturity at T and strikes $k \in \mathbb{K} \subseteq \mathcal{X}$. To simplify the exposition, we suppose that mid prices, instead of bid and ask prices, are directly observed. As is the case with empirical option markets, we consider the case where $\mathbb{K}$ does not cover the full state space $\mathcal{X}$. Instead, there exists a minimal strike ℓ and a maximal strike u with $0 < \ell \leq 1 \leq u < \infty$ such that both $P(\ell) > 0$ and $C(u) > 0$. For the main part of this paper, we will make the convenient assumption that $\mathbb{K} = [\ell, u]$, i.e., that option prices are observed over a bounded interval of strikes. This allows us to isolate the consequences of unobserved tail option prices, without the need to deal with other interfering effects. The simplifying assumptions imposed here are not essential and their relaxation even strengthens our main conclusions, at the cost of additional formal complexity.

[1] We exclude the left boundary to also deal with functions that are undefined at the origin, such as $\log R$. Equivalently, we could use $\mathcal{X} = [0, \infty)$ and continuously extend the function definition, e.g., using $\log R = -\infty$.

Accounting for the additional no-arbitrage pricing constraints introduced through observations in the option market, we define the set of admissible risk-neutral probability measures

$$\mathcal{B} := \left\{ \mathbb{Q} \in \mathcal{A} : C^{\mathbb{Q}}(k) = C(k) \text{ and } P^{\mathbb{Q}}(k) = P(k) \text{ for each } k \in \mathbb{K} \right\} ,$$

where $P^{\mathbb{Q}}(k) := \mathrm{E}^{\mathbb{Q}}[(k-R)^+]$ and $C^{\mathbb{Q}}(k) := \mathrm{E}^{\mathbb{Q}}[(R-k)^+]$ denote the put and call prices, respectively, associated to $\mathbb{Q}$. The size, or volume, of $\mathcal{B}$ is intimately related to the degree of market incompleteness in the economy. In the extreme case where a continuum of option prices becomes observable for every strike $k \in \mathcal{X} = (0, \infty)$, the set $\mathcal{B}$ becomes a singleton, i.e., a unique $\mathbb{Q}$ exists that is compatible with observed option prices. In general, however, $\mathbb{Q}$ is not uniquely pinned down by the imposed pricing constraints. With the choice $\mathbb{K} = [\ell, u]$ and the assumption that $P(\ell) > 0$ and $C(u) > 0$, the only indeterminacy in fact concerns the tail behavior of $\mathbb{Q}$ in the intervals $(0, \ell]$ and $[u, \infty)$. In the unrealistic and counterfactual case that either $P(\ell) = 0$ or $C(u) = 0$, the indeterminacy in the respective tail would be completely resolved.

Within this paper, we are interested in gauging the minimal and maximal value of a moment $\mathrm{E}^{\mathbb{Q}}[f(R)]$ across admissible probability measures $\mathbb{Q} \in \mathcal{B}$ for some payoff function $f : \mathcal{X} \to \mathbb{R}$. Economically, our objective thereby is the construction of sharp lower and upper bounds for the arbitrage-free valuations of a payoff f that are consistent with the information contained in the option market. In order to exclude pathological cases and at the same time not incur much loss of generality, we restrict payoff functions to the admissible set

$$\mathcal{F} := \left\{ \text{piecewise-continuous function } f \text{ on } \mathcal{X} \right\} .$$

By piecewise-continuous we specifically mean that any $f \in \mathcal{F}$ exhibits at most finitely many discontinuity points in the interior of $\mathcal{X}$, each with a jump of finite size. This eliminates the possibility of singularities in the interior of $\mathcal{X}$ and allows to focus the attention on the tail behavior. It should be noted that the definition of $\mathcal{F}$ does not mention any differentiability requirements. The set $\mathcal{F}$ is rich enough to contain essentially all payoffs of practical relevance.

Due to our assumption that option prices are observed on $\mathbb{K} = [\ell, u]$, it makes sense to split the problem into an interior moment $\mathcal{I}[f] := \mathrm{E}^{\mathbb{Q}}[f(R)\, \mathbb{1}(\ell < R < u)]$, which is uniquely determined from the observed option prices for all $\mathbb{Q} \in \mathcal{B}$ (cf. appendix B.4), as well as tail moments $\mathrm{E}^{\mathbb{Q}}[f(R)\, \mathbb{1}(R \le \ell)]$ and $\mathrm{E}^{\mathbb{Q}}[f(R)\, \mathbb{1}(R \ge u)]$. In this context, $\mathbb{1}(A)$ denotes the indicator function of event A.

What remains is the construction of lower and upper bounds for the tail moments. It is noteworthy that bounds in the left and right tails may be treated separately in our setting with $\mathbb{K} = [\ell, u]$. The optimal (i.e., tightest possible) lower and upper bounds are denoted by $\mathcal{L}^L[f]$ and $\mathcal{U}^L[f]$ for left-tail moments as well as $\mathcal{L}^R[f]$ and $\mathcal{U}^R[f]$ for right-tail moments. Jointly, these determine the lower and upper bounds $\mathcal{L}[f]$ and $\mathcal{U}[f]$ for moments $\mathrm{E}^{\mathbb{Q}}[f(R)]$ across all admissible measures $\mathbb{Q} \in \mathcal{B}$ as in equations (3.1) and (3.2) in the upcoming result.

For functions $f \in \mathcal{F}$ that are convex in the respective tail, the following result moreover gives the bounds for tail moments in simple closed form. Notably, these bounds are entirely determined by the observable behavior of option prices at strikes ℓ and u, which is summarized by the option prices $P(\ell)$ and $C(u)$ as well as their (one-sided) derivatives $\partial_{k+}P(\ell)$ and $\partial_{k-}C(u)$.[2] Straightforwardly, the stated expressions also determine bounds for concave $f \in \mathcal{F}$, noting that $-f$ is then convex, which effectively

[2]When $P(k)$ and $C(k)$ are observed on the entire interval $[\ell, u]$, then $\partial_{k+}P(\ell)$ and $\partial_{k-}C(u)$ are "observable" via the usual limiting relations, $\partial_{k+}P(\ell) := \lim_{h \downarrow 0} \frac{P(\ell+h)-P(\ell)}{h}$ and $\partial_{k-}C(u) := \lim_{h \downarrow 0} \frac{C(u)-C(u-h)}{h}$.

flips the roles of lower and upper bound expressions. Finally, with some additional effort, we may also use these results to construct tractable bounds for general $f \in \mathcal{F}$, as discussed in appendix B.3 (cf. proposition B.4).

Result 3.1 (Moment bounds)**.** *For any $f \in \mathcal{F}$, lower and upper moment bounds are given by*

$$\mathcal{L}[f] := \inf_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)] = \mathcal{L}^L[f] + \mathcal{I}[f] + \mathcal{L}^R[f] \tag{3.1}$$

$$\mathcal{U}[f] := \sup_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)] = \mathcal{U}^L[f] + \mathcal{I}[f] + \mathcal{U}^R[f] \ . \tag{3.2}$$

Moreover, the following holds for left-tail and right-tail moments:

(i) If f is convex on $(0, \ell]$, then lower and upper left-tail moment bounds are given by

$$\mathcal{L}^L[f] := \inf_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)\,\mathbb{1}(R \le \ell)] = \partial_{k+}P(\ell)\, f\left(\ell - \frac{P(\ell)}{\partial_{k+}P(\ell)}\right) \tag{3.3}$$

$$\mathcal{U}^L[f] := \sup_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)\,\mathbb{1}(R \le \ell)] = \left(\partial_{k+}P(\ell) - \frac{P(\ell)}{\ell}\right) f(\ell) + \frac{P(\ell)}{\ell} \lim_{R\to 0} f(R) \ . \tag{3.4}$$

(ii) If f is convex on $[u, \infty)$, then lower and upper right-tail moment bounds are given by

$$\mathcal{L}^R[f] := \inf_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)\,\mathbb{1}(R \ge u)] = -\partial_{k-}C(u)\, f\left(u - \frac{C(u)}{\partial_{k-}C(u)}\right) \tag{3.5}$$

$$\mathcal{U}^R[f] := \sup_{\mathbb{Q}\in\mathcal{B}} \mathrm{E}^{\mathbb{Q}}[f(R)\,\mathbb{1}(R \le u)] = -\partial_{k-}C(u)\, f(u) + C(u) \lim_{R\to\infty} \frac{f(R)}{R} \ . \tag{3.6}$$

Besides its simplicity, computability, and ease of applicability, result 3.1 is noteworthy in particular for its estimates $\mathcal{L}[f]$ and $\mathcal{U}[f]$, which are superpositions of observed out-of-the-money put and call prices, exclusively. Intuition would prescribe that $\mathcal{L}[f] \ge \mathcal{L}^L[f] + \mathcal{I}[f] + \mathcal{L}^R[f]$ and $\mathcal{U}[f] \le \mathcal{U}^L[f] + \mathcal{I}[f] + \mathcal{U}^R[f]$, as left left-tail and right-tail bounds are obtained from separate optimization problems.

3.2 Robustness of option-implied moments

Per se, it is not assured that the moment bounds $\mathcal{L}[f]$ and $\mathcal{U}[f]$ are finite. For $f \in \mathcal{F}$ with convex tails, the upper bounds $\mathcal{U}^L[f]$ or $\mathcal{U}^R[f]$ become infinite whenever the tail behavior of f is such that the limits $\lim_{R\to 0} f(R)$ or $\lim_{R\to\infty} \frac{f(R)}{R}$ are infinite. Likewise, for $f \in \mathcal{F}$ with concave tails, the lower bounds $\mathcal{L}^L[f] = -\mathcal{U}^L[-f]$ or $\mathcal{L}^R[f] = -\mathcal{U}^R[-f]$ become infinite under equivalent conditions. Therefore, the above result 3.1 strongly prescribes an elementary notion of robustness with regards to payoffs.

Definition 3.1 (Robust payoffs)**.** *A payoff f is* robust *if both $\mathcal{L}[f]$ and $\mathcal{U}[f]$ are finite.*

For functions $f \in \mathcal{F}$ with convex or concave tails, it suffices by result 3.1 to consider the limits $\lim_{R\to 0} f(R)$ and $\lim_{R\to\infty} \frac{f(R)}{R}$ in order to qualify payoffs as robust or non-robust. For general functions in $\mathcal{F}$, these limits may not exist in the ordinary sense and more general conditions are required, as formalized in the following result. These conditions capture the same ideas of boundedness in the left tail and at most linear growth in the right tail.

Result 3.2 (Robust payoffs)**.** *A payoff $f \in \mathcal{F}$ is robust if and only if the following conditions are satisfied:*

(i) f is bounded in the left tail with $\limsup_{R\to 0} |f(R)| < \infty$,

(ii) f grows at most linearly in the right tail with $\limsup_{R\to\infty} \frac{|f(R)|}{R} < \infty$.

Robustness in the sense of definition 3.1 is a minimal property that should be asked of a payoff. It is a binary qualification and merely guarantees that lower and upper bounds are finite. However, it does not guarantee in any way that these bounds are narrow, i.e., that the spread $\mathcal{U}[f] - \mathcal{L}[f]$ is relatively close to zero. This latter property captures a stronger notion of robustness.

Definition 3.2 (Strongly robust payoffs)**.** *A payoff f is* strongly robust *if $\mathcal{U}[f] - \mathcal{L}[f]$ is small.*

Definition 3.2 allows to quantify different degrees of robustness using the explicit bounds in result 3.1, but at the same time remains somewhat vague. To provide a complete picture of the robustness properties of different payoffs in the remainder of this paper, we will investigate both the weak robustness property in definition 3.2 as well as quantify its strength in the sense of definition 3.2. The former is a purely theoretical property, while the latter requires data regarding the behavior of option prices at extreme strikes. For this purpose, we regularly use a realistically calibrated "average" option prices for the S&P 500 and, where considered instructive, also the actual time series of observed option prices. This option data is described in appendix C.

3.3 Robustness properties of some important return moments

As a first application of our theory in sections 3.1 and 3.2, we investigate several classes of moments that play an important role in the literature. In particular, we consider powers of returns in various forms, using the three classes of payoffs R^p, $(R-1)^p$, and $(\log R)^p$, i.e., powers of gross, net, and log returns. These classes cover the VIX and SVIX in examples 2.1 and 2.2 as special cases, up to scaling factors.

To not overload our exposition with formulas, we omit the presentation of explicit expressions for lower and upper bounds. For the most part, these expressions are direct applications of equations (3.3) to (3.6). Only for higher-order powers of log returns that do not satisfy the convexity (or concavity) requirements, we need to use a generalization of these formulas (cf. proposition B.4).

As it turns out, these classes of moments are generally not robust, except for few special cases. For powers of gross returns, which are unproblematic to define for any $p \in \mathbb{R}$, any powers $p > 1$ lead to superlinear growth in the right tail and, hence, to infinite upper bounds on their moments. In addition, any powers $p < 0$ correspond to unboundedness in the left tail, likewise resulting in infinite upper moment bounds. Only powers $0 \leq p \leq 1$ yield robust payoffs, as for these both lower and upper bounds are finite. For net returns, we restrict the definition to positive integer powers $p \in \mathbb{N}$ in order to avoid a singularity at $R = 1$. Except for the trivial choices $p = 0$ and $p = 1$ that are already pinned down by the basic no-arbitrage constraints in $\mathcal{A}$ and $\mathcal{B}$, any other higher-order powers also suffer from superlinear growth in the right tail, analogous to gross returns, and are thus as well associated to infinite upper moment bounds. For log returns, we restrict the definition again to positive integers $p \in \mathbb{N}$ to avoid a singularity at $R = 1$ and moreover exclude the trivial case $p = 0$. For any remaining powers $p > 0$, robustness issues arise due to the unboundedness of the payoff in the left tail. Depending on whether p is even or odd, either an infinite upper or lower bound results. The following result summarizes the discussed robustness properties of return moments.

Result 3.3. *The following properties of return moments hold:*

(i) $\boldsymbol{R^p}$**.** *For any* $p \in \mathbb{R}$,

$$\mathcal{L}[R^p] \geq 0 \ , \qquad \mathcal{U}[R^p] \begin{cases} = \infty & \textit{if } p < 0 \\ < \infty & \textit{if } 0 \leq p \leq 1 \\ = \infty & \textit{if } p > 1 \end{cases} . \tag{3.7}$$

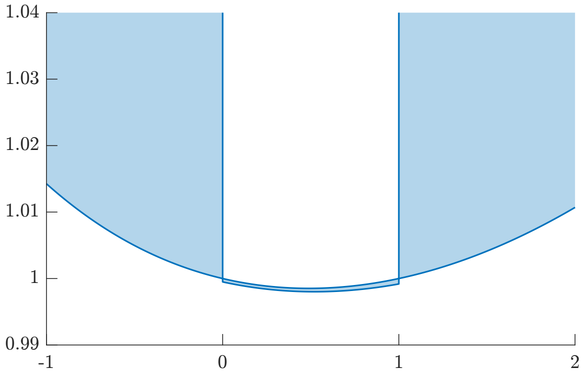
(ii) $\boldsymbol{(R-1)^p}$**.** *For any* $p \in \mathbb{N}$,

$$\mathcal{L}[(R-1)^p] \geq -1 \ , \qquad \mathcal{U}[(R-1)^p] \begin{cases} < \infty & \textit{if } p = 0, 1 \\ = \infty & \textit{if } p \geq 2 \end{cases} . \tag{3.8}$$

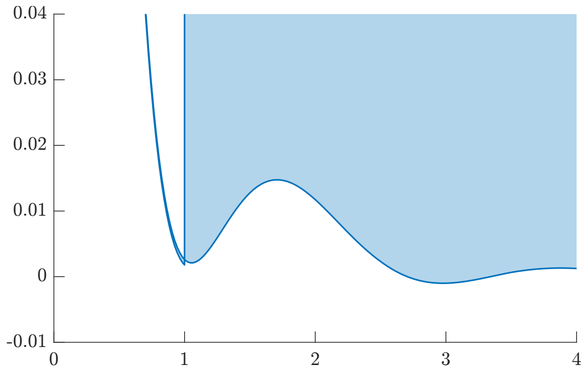
(iii) $\boldsymbol{(\log R)^p}$**.** *For any* $p \in \mathbb{N}$ *with* $p > 0$,

$$\mathcal{L}[(\log R)^p] \begin{cases} \geq 0 & \textit{if } p \textit{ even} \\ = -\infty & \textit{if } p \textit{ odd} \end{cases} , \qquad \mathcal{U}[(\log R)^p] \begin{cases} = \infty & \textit{if } p \textit{ even} \\ < \infty & \textit{if } p \textit{ odd} \end{cases} . \tag{3.9}$$

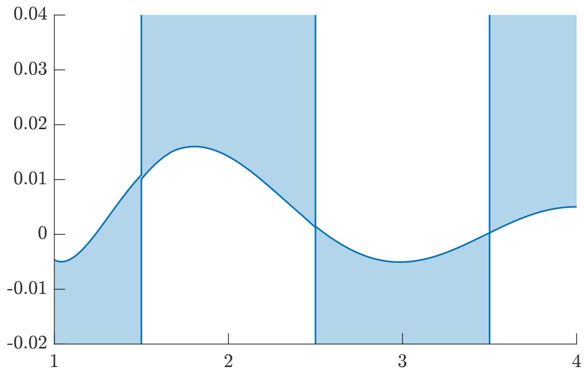
To further illustrate these robustness properties, figure 3.1 quantifies the model-free bounds for these return moments (extended to $p \in \mathbb{R}$), computed from average S&P 500 option prices. For gross returns R^p, option-implied moments are measured rather precisely for $0 \leq p \leq 1$, even perfectly exact at the end points of this interval (with values equal to one), while upper bounds become infinite when either $p < 0$ or $p > 0$. Due to the switch between convexity and concavity at $p = 0$ and $p = 1$, bounds are discontinuous in p at these points. For net returns $(R-1)^p$, we make similar observations in the range of positive powers p. Moment bounds are narrow for $0 \leq p \leq 1$, where the end points now take values equal to one at $p = 0$ and zero at $p = 1$, while upper bounds are infinite for any $p > 1$. For log returns $(\log R)^p$, it can be observed how the direction of unboundedness regularly flips, such that lower bounds are infinite at any odd integers and upper bounds are infinite at any even integers $p > 0$. The cosinusoidal behavior of moment bounds observed for both net and log returns is an artifact that stems from the extension from integer to real powers.



(a) R^p



(b) $(R-1)^p$



(c) $(\log R)^p$

Figure 3.1: Average bounds for return moments

This figure shows model-free bounds on option-implied return moments. Each plot shows lower and upper bounds $\mathcal{L}[f]$ and $\mathcal{U}[f]$ on option-implied moments (blue, y-axis), plotted against different powers p (x-axis). In panel (a), moments of gross returns use the payoff $f(R) = R^p$ for $p \in \mathbb{R}$. In panel (b), moments of net returns use the payoff $f(R) = (R-1)^p$, extended to $p \in \mathbb{R}$. In panel (c), moments of log returns use the payoff $f(R) = (\log R)^p$, extended to $p \in \mathbb{R}$. Bounds on option-implied moments are computed using average S&P 500 option prices at quarterly maturity.

4 Variance, semi-variance, and skewness

By our preceding analysis, common approaches to measure option-implied variance or skewness via simple powers of returns are non-robust. Specifically, non-robust option-implied measures include the VIX in CBOE (2009) as well as the SVIX in Carr and Corso (2001) and Martin (2017). In this section, we therefore develop a convenient framework to construct generalized variance, semi-variance, and skewness payoffs, building on power divergences as in Schneider and Trojani (2019). This framework provides a clear characterization of robust and non-robust payoffs in this domain. In particular, this section introduces a family of generalized variance payoffs, comprising the VIX^2 and SVIX^2 contracts as special cases, that carve out the set of robust variance payoffs with a single parameter.

4.1 Generalized (semi-)variance and skewness payoffs

Since for any pricing measure $\mathbb{Q} \in \mathcal{B}$ the no-arbitrage condition $\mathrm{E}^{\mathbb{Q}}[R] = 1$ holds, we work without loss of generality with the following family of power divergence payoffs:

$$D^p(R) := \begin{cases} \dfrac{R^p - p(R-1) - 1}{p(p-1)} & \text{if } p \neq 0, 1 \\ -\log R + R - 1 & \text{if } p = 0 \\ R \log R - R + 1 & \text{if } p = 1 \end{cases} . \tag{4.1}$$

By definition, power divergences span any power R^p of gross asset returns, except for the special cases at $p = 0$ and $p = 1$ that are defined by continuity. Further, any power divergence payoff for $p \in \mathbb{R}$ has locally the interpretation of a second moment payoff of log returns,

$$D^p(R) = \frac{1}{2}(\log R)^2 + o(|\log R|^2) . \tag{4.2}$$

As a consequence, any difference of two distinct power divergence payoffs also has locally the interpretation of a third realized moment of log returns since

$$SK^p(R) := \frac{1}{2p-1}[D^p(R) - D^{1-p}(R)] = \frac{1}{6}(\log R)^3 + o(|\log R|^3) \tag{4.3}$$

for any $p \geq 1/2$, where $SK^{1/2}(R) := \lim_{p \to 1/2} SK^p(R)$ is defined by continuity.

Very crucially for our purposes, the partly very different global behavior of power divergence payoffs $D^p(R)$ of different powers p can give rise to poor robustness properties in incomplete option markets for some of the associated option-implied moments. Such robustness properties are largely determined by the distinct behavior of these payoffs as $R \to 0$ and $R \to \infty$. In order to best accommodate their distinct left-tail and right-tail behavior, we consider in the sequel also lower and upper semi-power divergence payoffs, defined by

$$D^p_-(R) := D^p(R)\, \mathbb{1}(R < 1) , \qquad D^p_+(R) := D^p(R)\, \mathbb{1}(R \geq 1) . \tag{4.4}$$

Since the robustness properties of semi-power divergence payoffs intrinsically depend on their behavior as $R \to 0$ and $R \to \infty$, we characterize in more detail this tail behavior as follows:

$$\lim_{R \to 0} D^p_-(R) = \begin{cases} +\infty & \text{if } p \leq 0 \\ \dfrac{1}{p} & \text{if } 0 < p < 1 \\ \dfrac{1}{p} & \text{if } p \geq 1 \end{cases} , \qquad \lim_{R \to \infty} \frac{D^p_+(R)}{R} = \begin{cases} \dfrac{1}{1-p} & \text{if } p \leq 0 \\ \dfrac{1}{1-p} & \text{if } 0 < p < 1 \\ +\infty & \text{if } p \geq 1 \end{cases} . \tag{4.5}$$

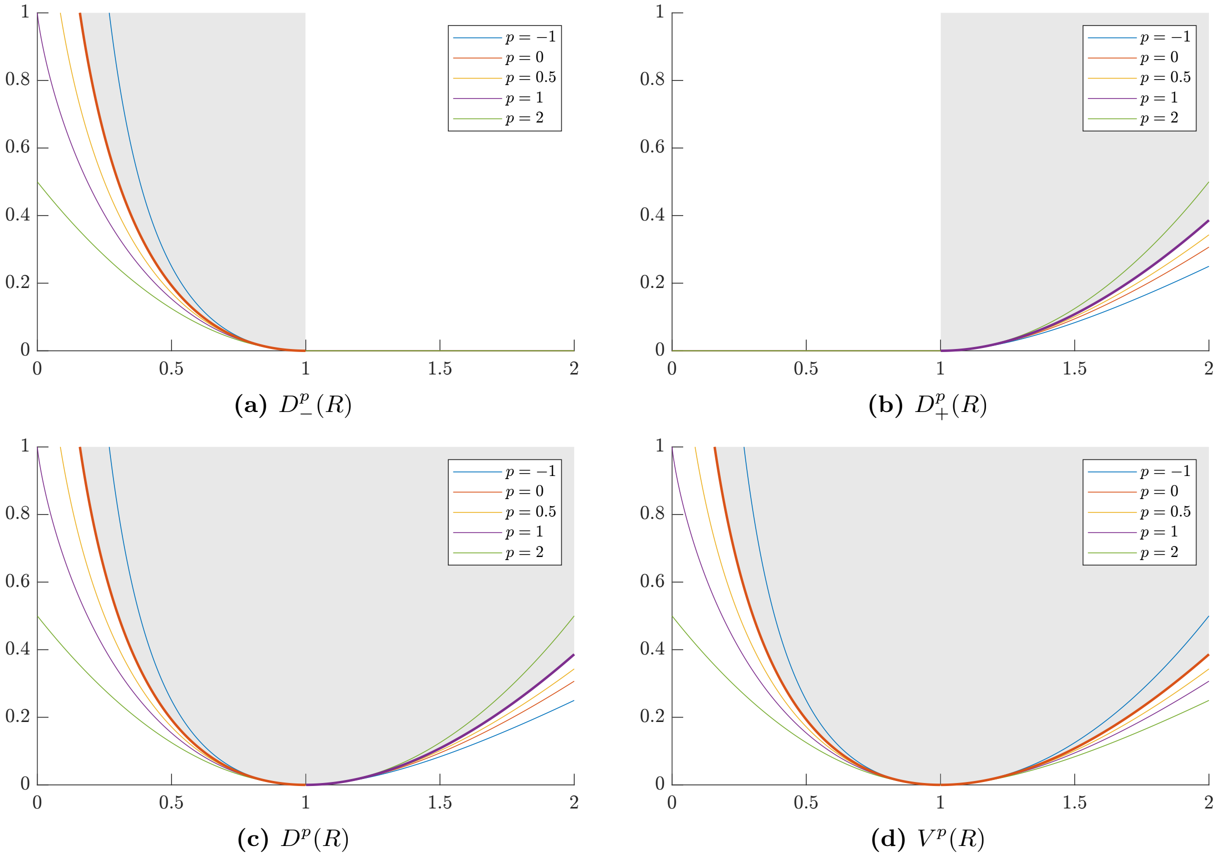
Lower semi-power divergence payoffs $D^p_-(R)$ of power $p \leq 0$ are unbounded in the left tail as $R \to 0$, while upper semi-power divergence payoffs $D^p_+(R)$ of power $p \geq 1$ exhibit superlinear growth in the right tail as $R \to \infty$. Therefore, in those regions they cannot be upper bounded by the payoff of a single position in an out-of-the-money option. This is the main reason why their option-implied moments necessarily have poor robustness properties in incomplete option markets. Importantly, all lower semi-power divergence payoffs of power $p > 0$ are bounded, while all upper semi-power divergence payoffs of power $p < 1$

have linear growth as $R \to \infty$. Therefore, the associated option-implied moments of these payoffs have improved robustness properties in incomplete option market.

The semi-power divergence payoffs $D^p_-(R)$ and $D^p_+(R)$ in equation (4.4) are the key building blocks for obtaining more robust notions of option-implied moments. To this end, note that by construction the sum of lower and upper semi-power divergence for an identical power aggregates to power divergence, i.e., a variance-like payoff according to equation (4.2). This property extends to sums of lower and upper semi-power divergences of different (complementary) powers and allows us to define, for any $p \in \mathbb{R}$, the symmetric variance payoff

$$V^p(R) := D^{1-p}_+(R) + D^p_-(R) = \frac{1}{2}(\log R)^2 + o(|\log R|^2) \ . \tag{4.6}$$

Figure 4.1 illustrates the main properties of semi-power divergence payoffs with respect to their global behavior for $R \to 0$ and $R \to \infty$. Panel (c) shows that power divergence payoffs $D^p(R)$ grow more rapidly as $R \to 0$ ($R \to \infty$) for decreasing powers p (increasing powers p). This feature is a direct consequence of the corresponding lower and upper semi-power divergence behavior of $D^p_-(R)$ and $D^p_+(R)$ depicted in panels (a) and (b). Panel (d) further shows that symmetric variance payoff $V^p(R)$ of power $p \geq 1/2$ ($p \leq 1/2$) are everywhere upper- (lower-) bounded by the associated power divergence payoff for the same power.



(a) $D^p_-(R)$ **(b)** $D^p_+(R)$ **(c)** $D^p(R)$ **(d)** $V^p(R)$

Figure 4.1: Divergence-based (semi-)variance payoffs

This figure shows payoffs of generalized (semi-)variances. Each plot shows payoff values (y-axis), plotted against gross returns R (x-axis) for different powers p. Shaded areas highlight non-robust powers. In panels (a) and (b), lower and upper semi-power divergences use the payoffs $f(R) = D^p_-(R)$ and $f(R) = D^p_+(R)$ in equation (4.4). In panel (c), power divergences use the payoff $f(R) = D^p(R)$ in equation (4.1). In panel (d), symmetric variances use the payoff $V^p(R)$ in equation (4.6).

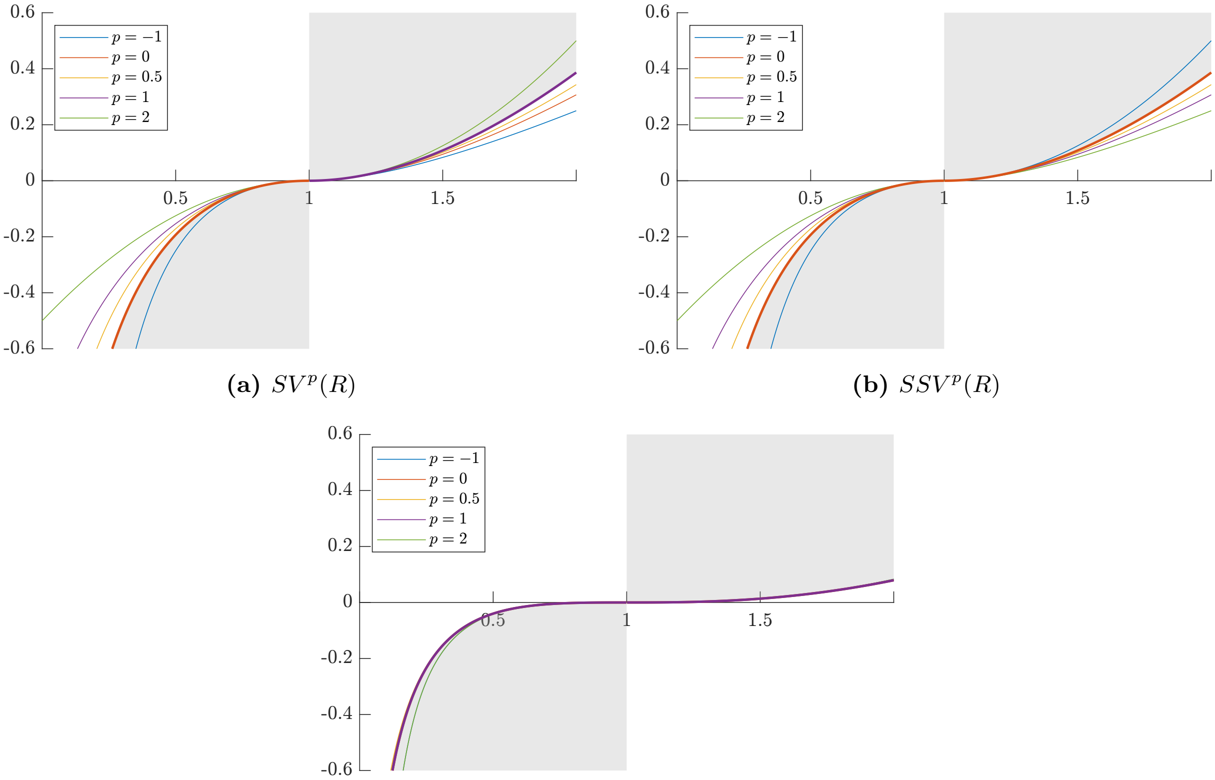
Analogously, note that the difference of upper and lower semi-power divergence is locally a payoff for trading upper against lower semi-variance. Choosing equal powers $p \in \mathbb{R}$ for both the lower and upper part yields the ordinary long-short semi-variance payoff

$$SV^p(R) := D^p_+(R) - D^p_-(R) = \frac{1}{2}(\log R)^2[\mathbb{1}(\log R \geq 0) - \mathbb{1}(\log R < 0)] + o(|\log R|^2)\ . \tag{4.7}$$

As for variance, the construction can be extended to differences of upper and lower semi-power divergences of different (complementary) powers. This yields, for any $p \in \mathbb{R}$, the symmetric long-short semi-variance payoff

$$SSV^p(R) := D^{1-p}_+(R) - D^p_-(R) = \frac{1}{2}(\log R)^2[\mathbb{1}(\log R \geq 0) - \mathbb{1}(\log R < 0)] + o(|\log R|^2)\ . \tag{4.8}$$

Figure 4.2 visualizes the long-short semi-power divergence and skewness payoffs, which directly inherit their main properties from the respective (semi-)power divergence payoffs. All payoffs are defined to be negative for $R < 1$ and positive for $R \geq 1$. In panel (a), the ordinary long-short semi-variance payoff $SV^p(R)$ decreases more rapidly in the left tail as $R \to 0$ for decreasing powers p, while it increases more rapidly in the right tail as $R \to \infty$ for increasing powers p. The symmetric long-short semi-variance payoff $SSV^p(R)$ in panel (b), however, grows slower simultaneously in both tails with increasing powers p. Finally, the skewness payoff $SK^p(R)$ in panel (c) apparently shows relatively variation within the plotted range, but exhibits distinctively different behavior in the extreme tails outside of the depicted return range.



(a) $SV^p(R)$ **(b)** $SSV^p(R)$ **(c)** $SK^p(R)$

Figure 4.2: Divergence-based long-short semi-variance and skewness payoffs

This figure shows payoffs of generalized long-short semi-variances and skewness. Each plot shows payoff values (y-axis), plotted against gross returns R (x-axis) for different powers p. Shaded areas highlight non-robust powers. In panel (a), long-short semi-variances use the payoff $f(R) = SV^p(R)$ in equation (4.7). In panel (b), symmetric long-short semi-variances use the payoff $f(R) = SSV^p(R)$ in equation (4.8). In panel (c), skewness uses the payoff $SK^p(R)$ in equation (4.3).

4.2 Robustness properties of generalized (semi-)variance and skewness

The key insight of the preceding discussion in section 4.1 is that in order to obtain variance-like and upper vs. lower semi-variance-like payoffs, one can equivalently make use of two distinct semi-power divergences having a different global behavior as $R \to 0$ and $R \to \infty$, respectively. This additional flexibility is the key for obtaining notions of option-implied moments with improved robustness properties using our main results 3.1 and 3.2.

By construction, moment bounds for the generalized (semi-)variance and skewness payoffs introduced in section 4.1 will directly depend on the moment bounds for semi-power divergences. Therefore, we first investigate the moment bounds and resulting robustness properties of these semi-power divergences. Due to the convexity of semi-power divergences $D^p_-(R)$ and $D^p_+(R)$ for any $p \in \mathbb{R}$, sharp lower and upper moment bounds immediately follow from result 3.1, applying equations (3.3) to (3.6). Once expressions for the moment bounds for semi-power divergences are available, moment bounds for divergences $D^p(R) =$

$D^p_-(R) + D^p_+(R)$ in our simplified setting satisfy convenient aggregation properties,

$$\mathcal{L}[D^p] = \mathcal{L}[D^p_-] + \mathcal{L}[D^p_+] \tag{4.9}$$

$$\mathcal{U}[D^p] = \mathcal{U}[D^p_-] + \mathcal{U}[D^p_+] \ . \tag{4.10}$$

Analogous relations can be exploited to obtain moment bounds for any of the generalized (semi-)variance and skewness payoffs, depending on $\mathcal{L}[D^p_-]$ and $\mathcal{U}[D^p_-]$ as well as $\mathcal{L}[D^p_+]$ and $\mathcal{U}[D^p_+]$.

Apart from closed-form expressions for moment bounds, our focus lies on the implied robustness properties, using result 3.2 in conjunction with the characterization of the tail behavior of (semi-)power divergences in equation (4.5). In particular, we observe that upper bounds for $D^p_-(R)$ are infinite for any $p \leq 0$, due to the unboundedness in the left tail, so that lower semi-power divergence payoffs are robust only if $p > 0$. Moreover, upper bounds for $D^p_+(R)$ are infinite for any $p \geq 1$, due to the superlinear growth in the right tail, implying that upper semi-power divergence payoffs are robust if $p < 1$. These robustness properties of semi-power divergences are summarized in the following result.

Result 4.1. *The following properties of semi-power divergences hold:*

(i) **Lower semi-power divergence.**

$$\mathcal{L}[D^p_-] \geq 0 \ , \qquad \mathcal{U}[D^p_-] \begin{cases} = \infty & \text{if } p \leq 0 \\ < \infty & \text{if } p > 0 \end{cases} . \tag{4.11}$$

(ii) **Upper semi-power divergence.**

$$\mathcal{L}[D^p_+] \geq 0 \ , \qquad \mathcal{U}[D^p_+] \begin{cases} < \infty & \text{if } p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases} . \tag{4.12}$$

By their construction together with result 4.1, variance-type payoffs $D^p(R)$ are robust exactly if both lower and upper semi-power divergence payoffs $D^p_-(R)$ and $D^p_+(R)$ of the same power p are robust. This is the case only for powers $0 < p < 1$, as otherwise upper bounds are infinite if either $p \leq 0$ or $p \geq 1$ due to the lower or upper semi-divergence component. In contrast, the symmetric variance payoff $V^p(R)$ combines wider ranges of powers over which both the lower and the upper semi-divergence are robust. Specifically, the payoff $V^p(R)$ is robust for any $p > 0$, as then both $D^p_-(R)$ and $D^{1-p}_+(R)$ have finite upper bounds. The robustness properties of (generalized) variance payoffs are stated in the following result.

Result 4.2. *The following properties of (generalized) variances hold:*

(i) **Divergence.**

$$\mathcal{L}[D^p] \geq 0 \ , \qquad \mathcal{U}[D^p] \begin{cases} = \infty & \text{if } p \leq 0 \\ < \infty & \text{if } 0 < p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases} . \tag{4.13}$$

(ii) **Symmetric variance.**

$$\mathcal{L}[V^p] \geq 0 \ , \qquad \mathcal{U}[V^p] \begin{cases} = \infty & \text{if } p \leq 0 \\ < \infty & \text{if } p > 0 \end{cases} . \tag{4.14}$$

The robustness properties of semi-power divergence payoffs in result 4.1 also directly translate to (generalized) semi-variance and skewness payoffs. For the ordinary long-short semi-variance $SV^p(R)$ and the skewness $SK^p(R)$, again only powers $0 < p < 1$ lead to robust payoffs. Instead, the symmetric long-short semi-variance payoffs $SSV^p(R)$ are robust for any $p > 0$. This leads to the robustness properties of (generalized) semi-variance and skewness payoffs in the following result.

Result 4.3. *The following properties of (generalized) semi-variances and skewness hold:*

(i) **Long-short semi-variance.**

$$\mathcal{L}[SV^p] \begin{cases} = -\infty & \text{if } p \leq 0 \\ > -\infty & \text{if } 0 < p < 1 \\ > -\infty & \text{if } p \geq 1 \end{cases}, \qquad \mathcal{U}[SV^p] \begin{cases} < \infty & \text{if } p \leq 0 \\ < \infty & \text{if } 0 < p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases}. \tag{4.15}$$

(ii) **Symmetric long-short semi-variance.**

$$\mathcal{L}[SSV^p] \begin{cases} = -\infty & \text{if } p \leq 0 \\ > -\infty & \text{if } p > 0 \end{cases}, \qquad \mathcal{U}[SSV^p] \begin{cases} = \infty & \text{if } p \leq 0 \\ < \infty & \text{if } p > 0 \end{cases}. \tag{4.16}$$

(iii) **Skewness.**

$$\mathcal{L}[SK^p] \begin{cases} = -\infty & \text{if } p \leq 0 \\ > -\infty & \text{if } 0 < p < 1 \\ = -\infty & \text{if } p \geq 1 \end{cases}, \qquad \mathcal{U}[SK^p] \begin{cases} = \infty & \text{if } p \leq 0 \\ < \infty & \text{if } 0 < p < 1 \\ = \infty & \text{if } p \geq 1 \end{cases}. \tag{4.17}$$

Overall, results 4.1 to 4.3 leave a key message. For ordinary variance-type payoffs $D^p(R)$, only a small region of robust powers p remains, which excludes most common option-implied measures. The VIX^2 contract in CBOE (2009), corresponding to the special case with $p = 0$, is a boundary case that is just outside the lower part of the robustness region. The corresponding boundary case on the upper part of the robustness region is the Kullback-Leibler payoff at $p = 1$, which is used in Gao and Martin (2021). The SVIX^2 contract in Carr and Corso (2001) and Martin (2017), corresponding to the choice $p = 2$, is not even a boundary case from the perspective of its robustness properties. Even when choosing robust powers $0 < p < 1$, one should not expect particularly strong robustness properties because each of these powers is located rather close to the boundaries. The symmetric variance payoff $V^p(R)$ considerably strengthens the robustness properties by allowing to simultaneously move away from the boundary of the robustness region in both tails. Analogous ideas underlie the relation between ordinary semi-variance payoffs $SV^p(R)$ and the symmetric semi-variance payoffs $SSV^p(R)$. Figure 4.3 illustrates the robustness regions for the different power divergence-based payoffs.

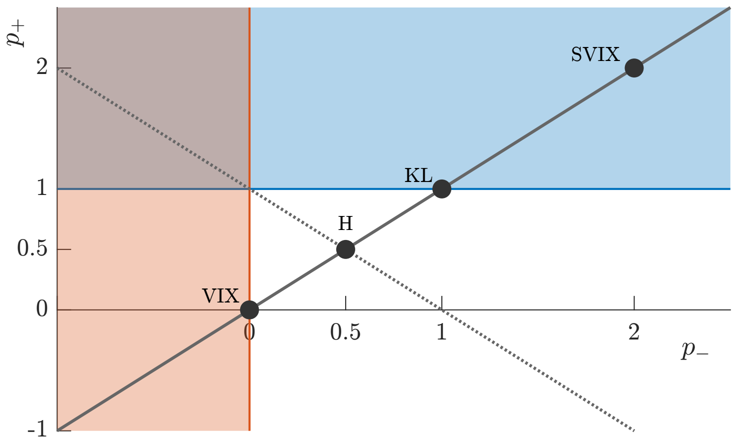


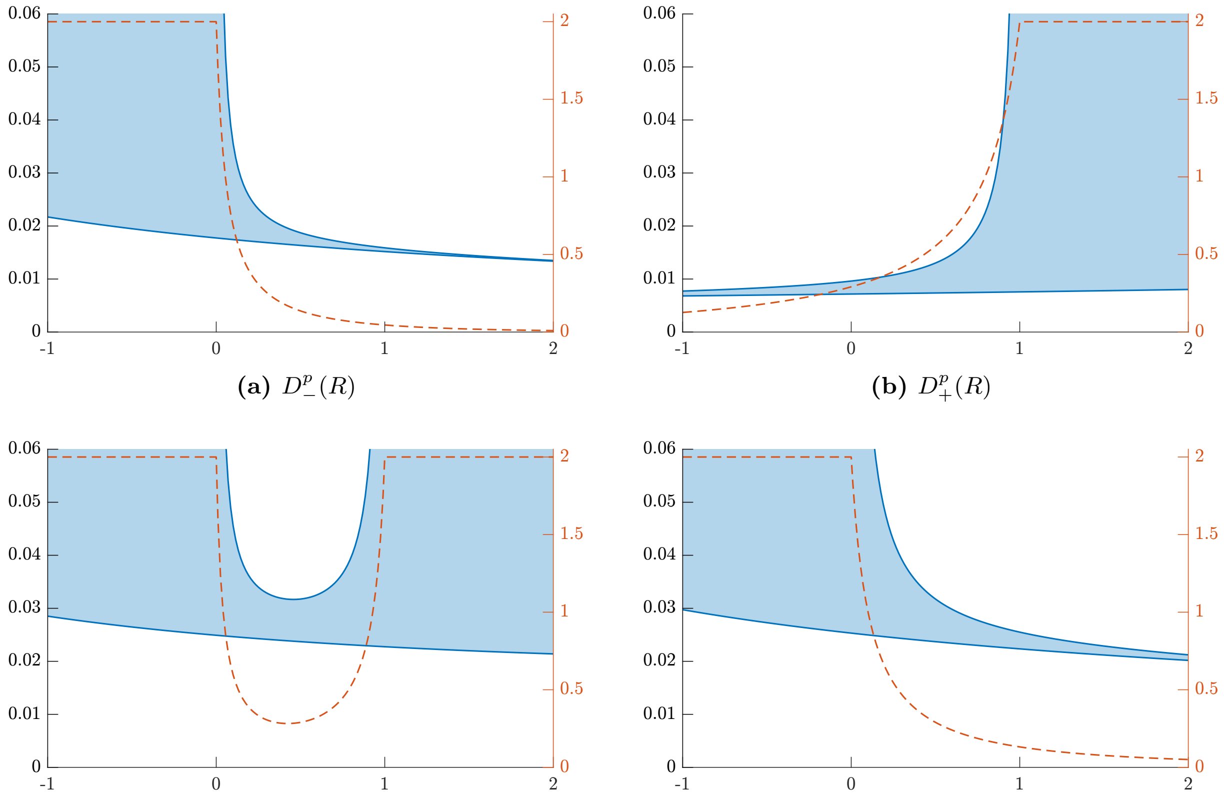
Figure 4.3: Robustness region for power divergence-based payoffs

This figure shows the robustness region of generalized measures of the form $D_+^{p_+}(R) \pm D_-^{p_-}(R)$. The red shaded area with boundary $p_- = 0$ highlights non-robust powers due to left-tail behavior. The blue shaded area with boundary $p_+ = 1$ highlights non-robust powers due to right-tail behavior. The diagonal solid line represents traditional measures. Important special cases are emphasized by dots: VIX, Hellinger (H), Kullback-Leibler (KL), SVIX. The anti-diagonal dotted line represents novel symmetric measures.

4.3 Empirical bounds on generalized (semi-)variance and skewness

To provide a more detailed analysis of the different robustness levels achieved by the generalized variance, semi-variance, and skewness payoffs in section 4.1, we now investigate the model-free bounds obtained from average S&P 500 option prices.

Figure 4.4 shows these bounds for variance and semi-variance payoffs. For the lower semi-power divergence payoffs $D_-^p(R)$ in panel (a), we observe that lower and upper bounds are generally decreasing in p, with upper bounds diverging to infinity when approaching the boundary case $p = 0$ from the right. This behavior is mirrored by the upper semi-power divergence payoffs $D_+^p(R)$ in panel (b), whose lower and upper bounds are increasing in p. The upper bounds are now diverging to infinity for powers approaching the boundary case at $p = 1$ from the left. In relative terms, both $D_-^p(R)$ for larger p and $D_+^p(R)$ for smaller p can be measured rather precisely from observed option prices, with some asymmetries due to the wider strike ranges available for out-of-the-money puts compared to calls. Essentially combining the behavior of lower and upper semi-power divergences, the power divergence payoffs $D^p(R)$ in panel (c) feature a lower bound that is decreasing in p and an upper bound that is U-shaped and increases to infinity at both boundary cases $p = 0$ and $p = 1$. Even for the robust payoffs with $0 < p < 1$, the relative width of moment bounds remains quite substantial. The picture changes when considering the symmetric variance payoffs V^p in panel (d), which combine favorable regions where observed option prices imply precise measurements of both lower and upper semi-power divergences $D_-^p(R)$ and $D_+^{1-p}(R)$. What results is again a downward-sloping shape of the lower bound with increasing p, but also a monotonously decreasing upper bound that diverges to infinity when approaching $p = 0$ from the right. For sufficiently large powers p, risk-neutral expectations of symmetric variance payoffs $V^p(R)$ can be measured from observed option prices in a strongly robust way.

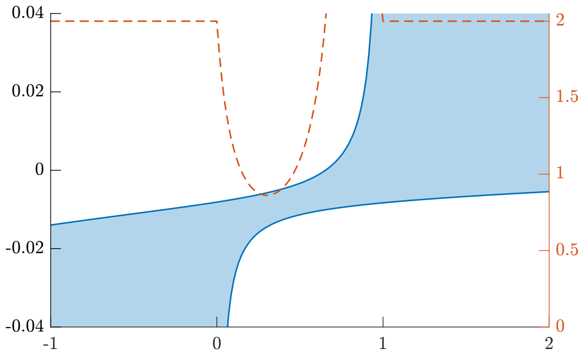


(a) $D_-^p(R)$ **(b)** $D_+^p(R)$ **(c)** $D^p(R)$ **(d)** $V^p(R)$

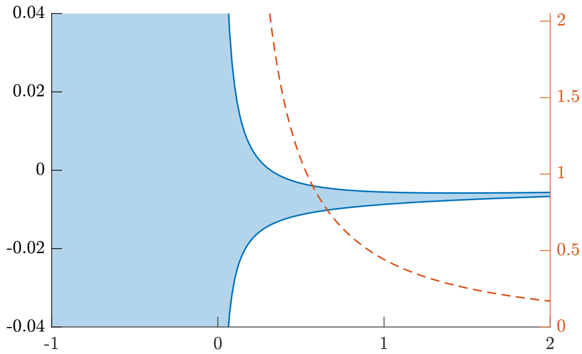
Figure 4.4: Average bounds for (semi-)variance payoffs

This figure shows model-free bounds on (annualized) option-implied generalized (semi-)variances. Each plot shows lower and upper bounds $\mathcal{L}[f]$ and $\mathcal{U}[f]$ on option-implied moments (blue, left y-axis) and relative widths $(\mathcal{U}[f]-\mathcal{L}[f])/(|\mathcal{U}[f]+\mathcal{L}[f]|/2)$ (red, right y-axis), plotted against different powers p (x-axis). In panels (a) and (b), lower and upper semi-power divergences use the payoffs $f(R) = D_-^p(R)$ and $f(R) = D_+^p(R)$ in equation (4.4). In panel (c), power divergences use the payoff $f(R) = D^p(R)$ in equation (4.1). In panel (d), symmetric variances use the payoff $V^p(R)$ in equation (4.6). Bounds on option-implied moments are computed using average S&P 500 option prices at quarterly maturity.

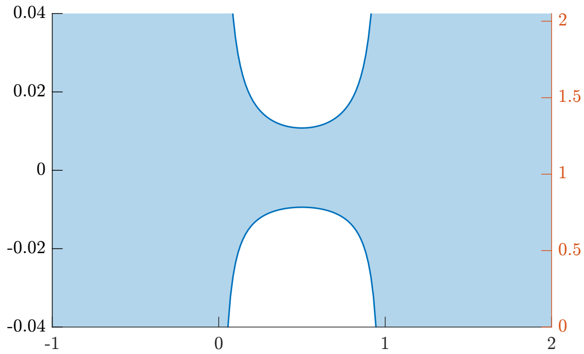
Figure 4.5 repeats the exercise for long-short semi-variance and skewness payoffs. In panel (a), we see that the ordinary long-short semi-variance payoffs $SV^p(R)$ again suffer from issues arising because they combine lower and upper semi-power divergences of the same power p. Both lower and upper bounds are upward-sloping, but simultaneously finite only for $0 < p < 1$. Even when both are finite, the relative width of moment bounds remains sizable. Unlike this, the symmetric long-short semi-variance payoffs $SSV^p(R)$ in panel (b) are able to achieve relatively narrow moment bounds when choosing sufficiently large p in regions where both lower semi-power divergence payoffs $D_-^p(R)$ and their upper counterparts $D_+^{1-p}(R)$ are measured precisely from observed option prices. Finally, panel (c) shows that for skewness payoffs SK^p, bounds are wide (outside the plotted range) even in the robust range of powers $0 < p < 1$.



(a) $SV^p(R)$



(b) $SSV^p(R)$



(c) $SK^p(R)$

Figure 4.5: Average bounds for long-short semi-variance and skewness payoffs

This figure shows model-free bounds on (annualized) option-implied generalized long-short semi-variances and skewness. Each plot shows lower and upper bounds $\mathcal{L}[f]$ and $\mathcal{U}[f]$ on option-implied moments (blue, left y-axis) and relative widths $(\mathcal{U}[f]-\mathcal{L}[f])/(|\mathcal{U}[f]+\mathcal{L}[f]|/2)$ (red, right y-axis), plotted against different powers p (x-axis). In panel (a), long-short semi-variances use the payoff $f(R) = SV^p(R)$ in equation (4.7). In panel (b), symmetric long-short semi-variances use the payoff $f(R) = SSV^p(R)$ in equation (4.8). In panel (c), skewness uses the payoff $SK^p(R)$ in equation (4.3). Bounds on option-implied moments are computed using average S&P 500 option prices at quarterly maturity.

5 Bounds on physical moments

The recent literature proposes several bounds on physical market return expectations based on option-implied risk-neutral moments. A lower bound for expected market gross returns based on the SVIX is proposed in Martin (2017), which is extended in Schneider and Trojani (2019) to a broader family of moment bounds. A similar bound for expected market log returns is suggested in Gao and Martin (2021). Applications of these bounds to study the cross-section of stock expected returns have also been recently considered (e.g., Bakshi et al. 2019; Chabi-Yo et al. 2023; Chabi-Yo and Loudis 2020; Kadan and Tang 2019; Martin and Wagner 2019). Due to the relevance of such option-implied bounds on physical return moments and their reliance on non-robust payoffs, this section proposes novel classes of robust bounds.

5.1 Generalized bounds on physical moments

Using robust symmetric variance payoffs and the widespread evidence of a negative market variance risk premium, we borrow from Schneider and Trojani (2019) to obtain a first robust option-implied upper bound on the expected market gross return.

Result 5.1. *Let* $\mathrm{E}[V^p(R)] < \infty$ *and the risk premium* $\mathrm{E}[V^p(R)] - \mathrm{E}^{\mathbb{Q}}[V^p(R)]$ *of the symmetric variance payoff* $V^p(R)$ *be negative for some* $\mathbb{Q} \in \mathcal{B}$ *and* $p > 0$*. It then follows that*

$$\mathrm{E}^{\mathbb{Q}}[V^p(R)] \geq \mathrm{E}[V^p(R)] \geq V^p(\mathrm{E}[R]) \ . \tag{5.1}$$

It particular, if $\mathrm{E}[R] \geq 1$*, it holds that*

$$(D_+^{1-p})^{-1}\big(\mathrm{E}^{\mathbb{Q}}[V^p(R)]\big) \geq \mathrm{E}[R] \ . \tag{5.2}$$

A distinct property of the upper bound (5.1) is that it is robust in incomplete option markets whenever $p > 0$. Analogous upper bounds based on power divergences of powers $p \notin (0, 1)$, such as part of those in Schneider and Trojani (2019), are instead not robust. Furthermore, for any $p \geq 1$, the robust upper bound (5.1) based on symmetric power divergence is tighter than any other robust upper bound based on power divergences.

To obtain robust option-implied moment lower bounds on market returns, we follow the literature (e.g., Gao and Martin 2021; Martin 2017) and introduce a suitable set of so-called Negative Covariance Conditions (NCC). For gross returns, extending Martin (2017), we define the following set of global conditions.

Definition 5.1. *Given the stochastic discount factor* $M^{\mathbb{Q}}$ *relative to some* $\mathbb{Q} \in \mathcal{B}$*, we say that global condition* NCC(p) *holds for* $\mathbb{Q}$ *if*

$$\mathrm{Cov}[M^{\mathbb{Q}} R^p, R] \leq 0 \ . \tag{5.3}$$

For instance, global condition NCC(p) holds in any economy where the SDF projection is of power type for a sufficiently large relative risk aversion, i.e., $M \propto R^{-\gamma}$ for some $\gamma \geq p \geq 0$. Martin (2017) specifically uses global condition NCC(1) to derive a lower bound for the expected market gross return.

Result 5.2. *If global condition* NCC(1) *holds for some* $\mathbb{Q} \in \mathcal{B}$*, then for any* $p \geq 2$*,*

$$\mathrm{E}[R] \geq 1 + 2\,\mathrm{E}^{\mathbb{Q}}[D^2(R)] \geq 1 + 2\,\mathrm{E}^{\mathbb{Q}}[V^p(R)] \ . \tag{5.4}$$

The first inequality in equation (5.4) corresponds to the lower bound proposed by Martin (2017), which builds on the SVIX-type payoff $D^2(R)$. The second inequality in equation (5.4) further introduces more conservative lower bounds based on the symmetric power divergence payoffs $V^p(R)$ for $p \geq 2$. As these lower bounds are decreasing in p for any given $\mathbb{Q}$, the tightest of such bounds in the family of symmetric power divergences under global condition NCC(1) is attained at $p = 2$. In setting with incomplete option markets, the lower bounds based on symmetric power divergence are robust, whereas the Martin (2017) lower bound is non-robust.

A similar construction yields lower bounds on the expected market log return. Generalizing the setup in Gao and Martin (2021), we introduce the following set of global and local conditions.

Definition 5.2. *Given the stochastic discount factor* $M^{\mathbb{Q}}$ *relative to some* $\mathbb{Q} \in \mathcal{B}$*, we say that global condition* logNCC(p) *holds for* $\mathbb{Q}$ *if*

$$\mathrm{Cov}[M^{\mathbb{Q}} R^p, \log R] \leq 0 \ . \tag{5.5}$$

Moreover, we say that local condition $\text{logNCC}_-(p)$ *holds for* $\mathbb{Q}$ *if*

$$\mathrm{Cov}[M^{\mathbb{Q}} R^p, \mathbb{1}(R < 1) \log R] \leq 0 \ , \tag{5.6}$$

and that local condition $\text{logNCC}_+(p)$ *holds for* $\mathbb{Q}$ *if*

$$\mathrm{Cov}[M^{\mathbb{Q}} R^p, \mathbb{1}(R \geq 1) \log R] \leq 0 \ . \tag{5.7}$$

Global condition logNCC(1) in equation (5.5) is used in Gao and Martin (2021) to motivate a corresponding lower bound on expected market log returns. The key distinction between local and global logNCC conditions pertains the potentially different local SDF co-movement properties with log returns in regions of positive/negative returns. For instance, global condition logNCC(p) and local conditions $\text{logNCC}_-(p)$ and $\text{logNCC}_+(p)$ hold for a power-type SDF projection with a sufficiently large relative risk aversion, i.e., $M \propto R^{-\gamma}$ for some $\gamma \geq p \geq 0$. This is a benchmark setting where the SDF global and local co-movement properties with returns coincide by construction. Conversely, local logNCC conditions are suited for handing also settings where the SDF co-movement properties with log returns in regions of positive/negative returns may not be symmetric, such as, e.g., SDFs in economies with loss aversion. Among local logNCC conditions, we are primarily interested in conditions $\text{logNCC}_-(1)$ and $\text{logNCC}_+(0)$ because they provide robust lower bounds on physical expected log returns in incomplete option markets from the option-implied moments of associated semi-power divergences.

Result 5.3. *If global condition* logNCC(1) *holds for some* $\mathbb{Q} \in \mathcal{B}$*, then for any* $p \geq 1$*,*

$$\mathrm{E}[\log R] \geq \mathrm{E}^{\mathbb{Q}}[D^1(R)] \geq \mathrm{E}^{\mathbb{Q}}[V^p(R)] \ . \tag{5.8}$$

The first inequality in equation (5.8) reproduces the Gao and Martin (2021) lower bound on expected log returns. The second inequality in equation (5.8) provides more conservative lower bounds based on symmetric power divergence payoffs $V^p(R)$ for $p \geq 1$. The boundary case with $p = 1$ yields the tightest such lower bound for a given $\mathbb{Q}$ that can be attained by a symmetric power divergence under condition logNCC(1). The distinct property between the two types of bounds is that the latter is obtained from a robust option-implied moment in incomplete option markets, while the former Gao and Martin (2021) lower bound is not.

We next consider lower bounds on expected log returns deriving from local logNCC assumptions. In this way, we achieve several goals at the same time. First, we separately lower bound the first moment of log returns in regions of negative and positive returns, respectively. Second, the sum of these local bounds naturally aggregates to a global lower bound, which relies on suppler assumptions regarding the SDF co-movement properties with positive and negative returns, respectively. Third, some of these local bounds only depend on a set of option-implied moments that are robust in incomplete option markets.

Result 5.4. *The following implications hold:*

(i) If local condition $\log\text{NCC}_{-}(1)$ *holds for some* $\mathbb{Q} \in \mathcal{B}$*, then*

$$\mathrm{E}[\mathbb{1}(R < 1) \log R] \geq -P(1) + \mathrm{E}^{\mathbb{Q}}[D^1_{-}(R)] \ . \tag{5.9}$$

(ii) If local condition $\log\text{NCC}_{+}(0)$ *holds for some* $\mathbb{Q} \in \mathcal{B}$*, then*

$$\mathrm{E}[\mathbb{1}(R \geq 1) \log R] \geq C(1) - \mathrm{E}^{\mathbb{Q}}[D^0_{+}(R)] \ . \tag{5.10}$$

Therefore, if both local conditions $\log\text{NCC}_{-}(1)$ *and* $\log\text{NCC}_{+}(0)$ *hold for some* $\mathbb{Q} \in \mathcal{B}$*, then*

$$\mathrm{E}[\log R] \geq -\mathrm{E}^{\mathbb{Q}}[SSV^1(R)] \ . \tag{5.11}$$

The distinct structure of the different lower bounds in result 5.4 are a direct consequence of the different local logNCC conditions assumed over regions of positive/negative returns. Note that all these lower bounds are completely determined by two quantities alone: the price of an at-the-money option and an option-implied semi-power divergence of powers $p \in \{0, 1\}$. As a consequence, the robustness properties of these bounds in incomplete option markets are fully determined by those of the associated semi-power divergences appearing in the bounds. In particular, lower bounds depending on divergences $D^1_{-}(R)$ or $D^0_{+}(R)$ are robust in incomplete option markets, but bounds depending on divergence $D^0_{-}(R)$ are not.

5.2 Robustness properties of bounds on physical moments

Section 5.1 constructs an array of different lower bounds $\hat{\mu}^{\mathbb{Q}}$ and upper bounds $\check{\mu}^{\mathbb{Q}}$ for physical expectations of gross and log returns, each depending on some risk-neutral probability measure $\mathbb{Q} \in \mathcal{B}$ through NCCs or other assumptions placed on risk premia. In incomplete option markets, the bounds $\hat{\mu}^{\mathbb{Q}}$ and $\check{\mu}^{\mathbb{Q}}$ are generally not uniquely determined, but themselves allow for a range of possible values. If the relevant assumptions hold for a particular (unknown) $\mathbb{Q} \in \mathcal{B}$, then physical return expectations can only be bounded from below by the model-free lower bound $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ over all $\mathbb{Q} \in \mathcal{B}$, i.e., the most conservative of all feasible bounds. The distance to the corresponding model-free upper bound $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$ reflects the uncertainty pertaining to the physical return bound $\hat{\mu}^{\mathbb{Q}}$.[3] For upper bounds $\check{\mu}^{\mathbb{Q}}$, we analogously bound physical return expectations from above by the model-free upper bound $\mathcal{U}[\check{\mu}^{\mathbb{Q}}]$ over all $\mathbb{Q} \in \mathcal{B}$, where now the associated model-free lower bound $\mathcal{L}[\check{\mu}^{\mathbb{Q}}]$ measures the indeterminacy of the bound. Accordingly, it is reasonable to assess bounds on physical returns using the robustness criteria in section 3.2.

The required model-free lower and upper bounds can immediately be connected to the bounds established in section 4. Regarding the lower and upper bounds on expected gross returns in results 5.1 and 5.2, the following result directly makes the connection to the bounds on option-implied power divergence and symmetric variance payoffs in result 4.2. As a consequence, we find that Martin (2017)'s SVIX-based lower bound is non-robust, while the lower and upper bounds based on symmetric variance are robust.

[3] If one were to make the stronger assumption that the conditions underlying a physical return bound hold for all $\mathbb{Q} \in \mathcal{B}$ simultaneously, then it would immediately follow that the physical return expectation is bounded from below by $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$.

Result 5.5. *The following properties hold for the bounds on expected gross returns in results 5.1 and 5.2:*

(i) For the upper bound $\check{\mu}^{\mathbb{Q}} := (D_+^{1-p})^{-1}\big(\mathrm{E}^{\mathbb{Q}}[V^p(R)]\big)$ *with* $p > 0$ *in equation (5.2), it holds that*

$$\mathcal{L}[\check{\mu}^{\mathbb{Q}}] = (D_+^{1-p})^{-1}\big(\mathcal{L}[V^p]\big) \geq 1 \ , \qquad \mathcal{U}[\check{\mu}^{\mathbb{Q}}] = (D_+^{1-p})^{-1}\big(\mathcal{U}[V^p]\big) < \infty \ . \tag{5.12}$$

(ii) For the lower bound $\hat{\mu}^{\mathbb{Q}} := 1 + 2\,\mathrm{E}^{\mathbb{Q}}[D^2(R)]$ *in equation (5.4), it holds that*

$$\mathcal{L}[\hat{\mu}^{\mathbb{Q}}] = 1 + 2\,\mathcal{L}[D^2] \geq 1 \ , \qquad \mathcal{U}[\hat{\mu}^{\mathbb{Q}}] = 1 + 2\,\mathcal{U}[D^2] = \infty \ . \tag{5.13}$$

(iii) For the lower bound $\hat{\mu}^{\mathbb{Q}} := 1 + 2\,\mathrm{E}^{\mathbb{Q}}[V^p(R)]$ *with* $p \geq 2$ *in equation (5.4), it holds that*

$$\mathcal{L}[\hat{\mu}^{\mathbb{Q}}] = 1 + 2\,\mathcal{L}[V^p] \geq 1 \ , \qquad \mathcal{U}[\hat{\mu}^{\mathbb{Q}}] = 1 + 2\,\mathcal{U}[V^p] < \infty \ . \tag{5.14}$$

In a similar way, the next result also provides explicit expressions for the lower bounds on expected log returns in results 5.3 and 5.4. The bounds on option-implied power divergence and symmetric variance payoffs in result 4.2 likewise yield with regards to result 5.3 that the Gao and Martin (2021) bound is non-robust, while the bounds based on symmetric variance are robust. Moreover, the localized bounds in result 5.4 are also robust, using the bounds on semi-power divergence and symmetric semi-variance payoffs established in results 4.1 and 4.3.

Result 5.6. *The following properties hold for the bounds on expected log returns in results 5.3 and 5.4:*

(i) For the lower bound $\hat{\mu}^{\mathbb{Q}} := \mathrm{E}^{\mathbb{Q}}[D^1(R)]$ *in equation (5.8), it holds that*

$$\mathcal{L}[\hat{\mu}^{\mathbb{Q}}] = \mathcal{L}[D^1] \geq 0 \ , \qquad \mathcal{U}[\hat{\mu}^{\mathbb{Q}}] = \mathcal{U}[D^1] = \infty \ . \tag{5.15}$$

(ii) For the lower bound $\hat{\mu}^{\mathbb{Q}} := \mathrm{E}^{\mathbb{Q}}[V^p(R)]$ *with* $p \geq 1$ *in equation (5.8), it holds that*

$$\mathcal{L}[\hat{\mu}^{\mathbb{Q}}] = \mathcal{L}[V^p] \geq 0 \ , \qquad \mathcal{U}[\hat{\mu}^{\mathbb{Q}}] = \mathcal{U}[V^p] < \infty \ . \tag{5.16}$$

(iii) For the left-tail lower bound $\hat{\mu}_-^{\mathbb{Q}} := -P(1) + \mathrm{E}^{\mathbb{Q}}[D_-^1(R)]$ *in equation (5.9), it holds that*

$$\mathcal{L}[\hat{\mu}_-^{\mathbb{Q}}] = -P(1) + \mathcal{L}[D_-^1] \geq -1 \ , \qquad \mathcal{U}[\hat{\mu}_-^{\mathbb{Q}}] = -P(1) + \mathcal{U}[D_-^1] < \infty \ . \tag{5.17}$$

(iv) For the right-tail lower bound $\hat{\mu}_+^{\mathbb{Q}} := C(1) - \mathrm{E}^{\mathbb{Q}}[D_+^0(R)]$ *in equation (5.10), it holds that*

$$\mathcal{L}[\hat{\mu}_+^{\mathbb{Q}}] = C(1) - \mathcal{U}[D_+^0] > -\infty \ , \qquad \mathcal{U}[\hat{\mu}_+^{\mathbb{Q}}] = C(1) - \mathcal{L}[D_+^0] \leq 1 \ . \tag{5.18}$$

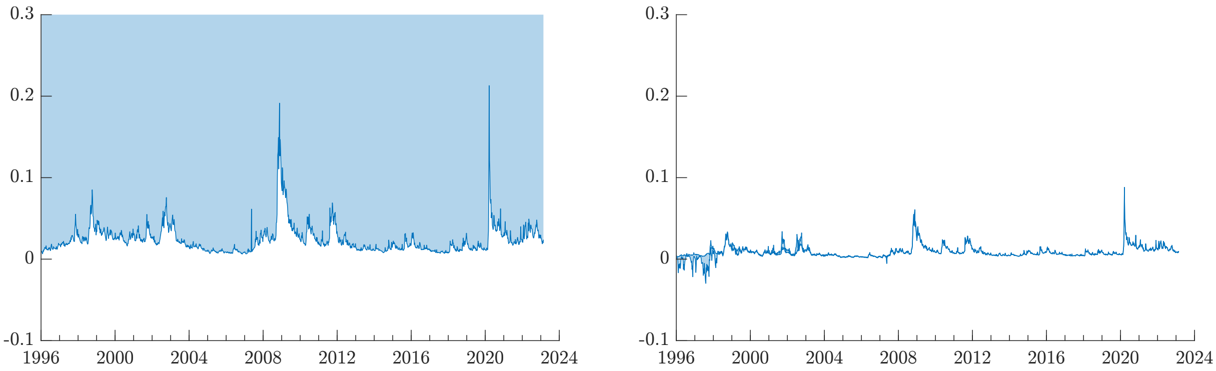
(v) For the lower bound $\hat{\mu}^{\mathbb{Q}} := -\mathrm{E}^{\mathbb{Q}}[SSV^1(R)]$ *in equation (5.11), it holds that*

$$\mathcal{L}[\hat{\mu}^{\mathbb{Q}}] = -\mathcal{U}[SSV^1] > -\infty \ , \qquad \mathcal{U}[\hat{\mu}^{\mathbb{Q}}] = -\mathcal{L}[SSV^1] < \infty \ . \tag{5.19}$$

5.3 Empirical bounds on physical moments

To make an illustration of the distinct behavior of non-robust and robust lower bounds on physical return expectations, we consider the time series behavior of model-free bounds for the Gao and Martin

(2021) lower bound in equation (5.8) and the robust lower bound in equation (5.11). Figure 5.1 shows the historical evolution of the associated model-free bounds $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ and $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$ according to result 5.6. Regarding the Gao and Martin (2021) bound, it should be noted that only the lower bound $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ is finite, while the upper bound $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$ is always infinite. By construction through a variance-type measure, $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ is moreover constrained to be always positive. In contrast, the robust bound in equation (5.11) has both $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ and $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$ finite, with occasionally negative values. Except for maybe the very early part of the sample period, due to narrower observed strike ranges then, the bound in equation (5.11) can be measured precisely from observed option prices, as the difference $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}] - \mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ is small. Overall, the bound in equation (5.11) is lower and less volatile compared to the Gao and Martin (2021) bound.



(a) Gao and Martin (2021) bound

(b) $\log\text{NCC}_{-}(1) + \log\text{NCC}_{+}(0)$ bound

Figure 5.1: Historical bounds on expected log returns

This figure shows the time series of model-free ranges of lower bounds on (annualized) physical log return expectations at quarterly time horizon. Each plot shows lower and upper bounds $\mathcal{L}[\hat{\mu}^{\mathbb{Q}}]$ and $\mathcal{U}[\hat{\mu}^{\mathbb{Q}}]$ on physical return bounds (blue, y-axis), plotted against time (x-axis). Panel (a) uses for $\hat{\mu}^{\mathbb{Q}}$ the Gao and Martin (2021) lower bound in equation (5.8). Panel (b) uses for $\hat{\mu}^{\mathbb{Q}}$ the robust lower bound in equation (5.11). Bounds on option-implied moments are computed weekly using S&P 500 option prices at quarterly maturity.

6 Conclusion

We assess the extent to which option portfolios are informative about risk-neutral moments. This question is motivated by numerous studies that exploit the conditional nature of risk-neutral moments to make predictions about subjective, or physical moments. We find that for certain kinds of commonly used variance swaps contracts for this purpose, underlying for instance the calculations of VIX and SVIX, option portfolios can not exclude the possibility that their prices are infinite.

Our investigations are based on rather generic and general assumptions, such as the absence of arbitrage and piecewise-continuous payoff functions, but nevertheless powerful enough to characterize the family of (non-)robust payoffs, whose prices can (not) possibly be infinite. We propose families of contracts that alleviate lack of robustness in extant definitions.

Replicating several empirical exercises from the literature that conjecture a link between risk-neutral and physical/natural moments, we illustrate the stark non-robustness. For instance, the lower bound on the equity premium proposed by Martin (2017) may well be very large or even infinite.

References

Bakshi, G., N. Kapadia, and D. Madan (2003). "Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options". *Review of Financial Studies* 16, pp. 101–143.

Bakshi, G. S., J. Crosby, X. Gao, and W. Zhou (2019). "A New Formula for the Expected Excess Return of the Market". workinpaper, Fox School of Business Research.

Banz, R. W. and M. H. Miller (1978). "Prices for State-Contingent Claims: Some Estimates and Applications". *The Journal of Business* 51 (4), pp. 653–672.

Bick, A. (1982). "Comments on the valuation of derivative assets". *Journal of Financial Economics* 10 (3), pp. 331–345.

Breeden, D. T. and R. H. Litzenberger (1978). "Prices of State-Contingent Claims Implicit in Option Prices". *Journal of Business* 51, pp. 621–651.

Britten-Jones, M. and A. Neuberger (2000). "Option Prices, Implied Price Processes, and Stochastic Volatility". *Journal of Finance* 55 (2), pp. 839–866.

Carr, P. and A. Corso (2001). "Covariance contracting for commodities". *Energy and Power Risk Management*, pp. 42–45.

Carr, P. and D. Madan (2001). "Towards a Theory of Volatility Trading". *Option Pricing, Interest Rates and Risk Management*. Ed. by E. Jouini, J. Cvitanic, and M. Musiela. Handbooks in Mathematical Finance. Cambridge University Press, pp. 458–476.

Carr, P. and L. Wu (2009). "Variance Risk Premiums". *Review of Financial Studies* 22, pp. 1311–1341.

CBOE (2009). *The CBOE volatility index - VIX*. CBOE.

Chabi-Yo, F., C. Dim, and G. Vilkov (2023). "Generalized Bounds on the Conditional Expected Excess Return on Individual Stocks". *Management Science* 69 (2), pp. 922–939.

Chabi-Yo, F. and J. Loudis (2020). "The conditional expected market return". *Journal of Financial Economics* 137 (3), pp. 752–786.

Davis, M., J. Obłój, and V. Raval (2014). "Arbitrage Bounds for Prices of Weighted Variance Swaps". *Mathematical Finance* 24, pp. 821–854.

Gao, C. and I. W. R. Martin (Aug. 2021). "Volatility, Valuation Ratios, and Bubbles: An Empirical Measure of Market Sentiment". *The Journal of Finance* 76 (6), pp. 3211–3254.

Hobson, D. and M. Klimmek (2012). "Model-independent hedging strategies for variance swaps". *Finance and Stochastics* 16, pp. 611–649.

Kadan, O. and X. Tang (July 2019). "A Bound on Expected Stock Returns". *The Review of Financial Studies* 33 (4), pp. 1565–1617.

Lee, R. W. (2004). "The Moment Formula for Implied Volatility at Extreme Strikes". *Mathematical Finance* 14 (3), pp. 469–480.

Martin, I. (2017). "What is the Expected Return on the Market?" *The Quarterly Journal of Economics* 132, p. 367.

Martin, I. W. R. and C. Wagner (2019). "What Is the Expected Return on a Stock?" *Journal of Finance* 74 (4), pp. 1887–1929.

Schneider, P. (2015). "Generalized risk premia". *Journal of Financial Economics* 116 (3), pp. 487–504.

Schneider, P. and F. Trojani (2019). "(Almost) Model-Free Recovery". *Journal of Finance* 74, pp. 323–370.

Appendix

A More on the introductory examples

This section provides further details on the introductory examples in section 2.

A.1 Piecewise-linear option price tails

Associated to the option prices in equation (2.2), we have a decomposition of the probability measure $\mathbb{Q}_{a,b}$ of the form $\mathbb{Q}_{a,b} = \mathbb{Q}_a^L + \mathbb{Q}^M + \mathbb{Q}_b^R$. The measure $\mathbb{Q}^M$ has support in (ℓ, u) and is fully determined by the observed option prices. In contrast, the tail measures $\mathbb{Q}_a^L$ and $\mathbb{Q}_b^R$ are supported in the respective tails $(0, \ell]$ and $[u, \infty)$ and are generally not identified. The left-tail measure $\mathbb{Q}_a^L$ is a discrete measure with mass located at a and ℓ,

$$\mathbb{Q}_a^L[R = a] = \frac{P(\ell)}{\ell - a} \,, \qquad \mathbb{Q}_a^L[R = \ell] = \partial_{k+}P(\ell) - \frac{P(\ell)}{\ell - a} \,. \tag{A.1}$$

The right-tail measure $\mathbb{Q}_b^R$ is a discrete measure with mass at u and b,

$$\mathbb{Q}_b^R[R = u] = \frac{C(u)}{b - u} - \partial_{k-}C(u) \,, \qquad \mathbb{Q}_b^R[R = b] = \frac{C(u)}{b - u} \,. \tag{A.2}$$

To guarantee positivity of both measures, the parameters a and b must satisfy the constraints $0 \leq a \leq \ell_* \leq \ell$ and $u \leq u_* \leq b < \infty$, respectively, where

$$\ell_* := \ell - \frac{P(\ell)}{\partial_{k+}P(\ell)} \,, \qquad u_* := u - \frac{C(u)}{\partial_{k-}C(u)} \,. \tag{A.3}$$

Except for pathological cases, it holds that $\ell_* > 0$.

For left-tail moments, note that we may write $\mathrm{E}^{\mathbb{Q}_{a,b}}[f(R)\, \mathbb{1}(R \leq \ell)] = \mathrm{E}^{\mathbb{Q}_a^L}[f(R)]$, which can be expressed as

$$\mathrm{E}^{\mathbb{Q}_a^L}[f(R)] = \underbrace{\left(\frac{P(\ell)}{\ell - a}\right)}_{\mathbb{Q}_a^L[R=a]} f(a) + \underbrace{\left(\partial_{k+}P(\ell) - \frac{P(\ell)}{\ell - a}\right)}_{\mathbb{Q}_a^L[R=\ell]} f(\ell) \,. \tag{A.4}$$

Analogously, for right-tail moments, we have that $\mathrm{E}^{\mathbb{Q}_{a,b}}[f(R)\, \mathbb{1}(R \geq u)] = \mathrm{E}^{\mathbb{Q}_b^R}[f(R)]$, which yields

$$\mathrm{E}^{\mathbb{Q}_b^R}[f(R)] = \underbrace{\left(\frac{C(u)}{b - u} - \partial_{k-}C(u)\right)}_{\mathbb{Q}_b^R[R=u]} f(u) + \underbrace{\left(\frac{C(u)}{b - u}\right)}_{\mathbb{Q}_b^R[R=b]} f(b) \,. \tag{A.5}$$

Since we may also write $\mathcal{P}^{\mathbb{Q}_{a,b}}[f] = \mathcal{P}^{\mathbb{Q}_a^L}[f]$ and $\mathcal{C}^{\mathbb{Q}_{a,b}}[f] = \mathcal{C}^{\mathbb{Q}_b^R}[f]$, the tail moments in equations (A.4) and (A.5) are related to the tail integrals in equations (2.3) and (2.4) via

$$\mathrm{E}^{\mathbb{Q}_a^L}[f(R)] = \mathcal{P}^{\mathbb{Q}_a^L}[f] + f(\ell)\, \partial_{k+}P(\ell) - f'(\ell)\, P(\ell) \tag{A.6}$$

$$\mathrm{E}^{\mathbb{Q}_b^R}[f(R)] = \mathcal{C}^{\mathbb{Q}_b^R}[f] - f(u)\, \partial_{k-}C(u) + f'(u)\, C(u) \,. \tag{A.7}$$

To interpret these expressions, it should be noted that $\mathrm{E}^{\mathbb{Q}_a^L}[1] = \partial_{k-}P(\ell)$ and $\mathrm{E}^{\mathbb{Q}_a^L}[R - \ell] = -P(\ell)$ as well as $\mathrm{E}^{\mathbb{Q}_b^R}[1] = -\partial_{k+}C(u)$ and $\mathrm{E}^{\mathbb{Q}_b^R}[R - u] = C(u)$. Effectively, $\mathcal{P}^{\mathbb{Q}_a^L}[f]$ and $\mathcal{C}^{\mathbb{Q}_b^R}[f]$ thus capture the nonlinear parts of the respective tail moments.

In the limit as $a \to 0$ and $b \to \infty$, the tail moments in equations (A.4) and (A.5) become

$$\lim_{a\to 0} \mathrm{E}^{\mathbb{Q}_a^L}[f(R)] = \frac{P(\ell)}{\ell} \left(\lim_{a\to 0} f(a) \right) + \left(\partial_{k+} P(\ell) - \frac{P(\ell)}{\ell} \right) f(\ell) \tag{A.8}$$

$$\lim_{b\to\infty} \mathrm{E}^{\mathbb{Q}_b^R}[f(R)] = -\partial_{k-} C(u)\, f(u) + C(u) \left(\lim_{b\to\infty} \frac{f(b)}{b} \right) \ . \tag{A.9}$$

With these explicit expressions for tail moments, we now revisit examples 2.1 and 2.2.

Example A.1 (VIX)**.** Using the payoff $\mathrm{VIX}^2(R) = -2\log R$ as in example 2.1, the left-tail moment $\mathrm{E}^{\mathbb{Q}_a^L}[\mathrm{VIX}^2(R)]$ is positive and increasing as $a \to 0$ as well as unbounded, since

$$\lim_{a\to 0} \mathrm{E}^{\mathbb{Q}_a^L}[\mathrm{VIX}^2(R)] = \frac{P(\ell)}{\ell} \left(\lim_{a\to 0} -2\log a \right) + \left(\partial_{k+} P(\ell) - \frac{P(\ell)}{\ell} \right) (-2\log \ell) = \infty \ .$$

The right-tail moment $\mathrm{E}^{\mathbb{Q}_b^R}[\mathrm{VIX}^2(R)]$ is negative and decreasing as $b \to \infty$, but bounded from below by

$$\lim_{b\to\infty} \mathrm{E}^{\mathbb{Q}_b^R}[\mathrm{VIX}^2(R)] = -\partial_{k-} C(u) \log u \ .$$

Consistent with our observations in example 2.1, VIX valuations $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{VIX}^2(R)]$ are thus bounded from below, but unbounded from above. ■

Example A.2 (SVIX)**.** For the payoff $\mathrm{SVIX}^2(R) = (R-1)^2$ as in example 2.2, the left-tail moment $\mathrm{E}^{\mathbb{Q}_a^L}[\mathrm{SVIX}^2(R)]$ is positive and increasing as $a \to 0$ with upper bound

$$\lim_{a\to 0} \mathrm{E}^{\mathbb{Q}_a^L}[\mathrm{SVIX}^2(R)] = \partial_{k+} P(\ell)\, (\ell - 1)^2 + P(\ell)\, (2 - \ell) \ .$$

The right-tail moment $\mathrm{E}^{\mathbb{Q}_b^R}[\mathrm{SVIX}^2(R)]$ is likewise positive and increasing as $b \to \infty$, but unbounded since

$$\lim_{b\to\infty} \mathrm{E}^{\mathbb{Q}_b^R}[\mathrm{SVIX}^2(R)] = -\partial_{k-} C(u)\, (u-1)^2 + C(u) \left(\lim_{b\to\infty} \frac{(b-1)^2}{b} \right) = \infty \ .$$

Consistent with our observations in example 2.2, SVIX valuations $\mathrm{E}^{\mathbb{Q}_{a,b}}[\mathrm{SVIX}^2(R)]$ are thus also bounded from below, but unbounded from above. ■

A.2 Power-law option price tails

Motivated by the insights of Lee (2004), consider a different parameterization $\mathbb{Q}_{q,p}$ defined through a decomposition of the form $\mathbb{Q}_{q,p} = \mathbb{Q}_q^L + \mathbb{Q}^M + \mathbb{Q}_p^R$ in terms of parameters $q \le 0$ and $p \ge 1$. Instead of the piecewise-linear tail behavior of option prices in equation (2.2), the tail measures $\mathbb{Q}_q^L$ and $\mathbb{Q}_p^R$ now prescribe a power-law behavior such that

$$P^{\mathbb{Q}_{q,p}}(k) = P(\ell) \left(\frac{k}{\ell} \right)^{1-q} \ , \qquad C^{\mathbb{Q}_{q,p}}(k) = C(u) \left(\frac{u}{k} \right)^{p-1} \ . \tag{A.10}$$

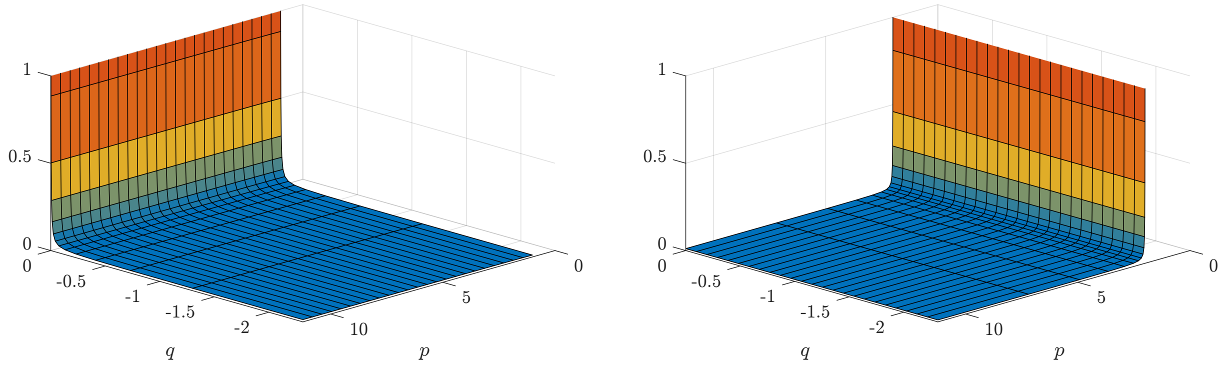
To ensure arbitrage-free extrapolations, the parameters need to satisfy the constraints $q_* \le q \le 0$ and $1 \le p \le p_*$, where

$$q_* := 1 - \frac{\partial_{k+} P(\ell)\, \ell}{P(\ell)} \ , \qquad p_* := 1 - \frac{\partial_{k-} C(u)\, u}{C(u)} \ . \tag{A.11}$$

With this specification, it holds that $q = \inf\{\alpha : \mathrm{E}^{\mathbb{Q}_{q,p}}[R^\alpha] < \infty\}$ and $p = \sup\{\alpha : \mathrm{E}^{\mathbb{Q}_{q,p}}[R^\alpha] < \infty\}$.

The boundary cases with $q = 0$ and $p = 1$ corresponds to the limiting cases of piecewise-linear tail option prices when $a \to 0$ and $b \to \infty$, respectively.

For a given payoff f, the tail integrals $\mathcal{P}^{\mathbb{Q}_{q,p}}[f]$ and $\mathcal{C}^{\mathbb{Q}_{q,p}}[f]$ may generally require the use of numerical integration techniques. In the case of the VIX^2 and SVIX^2 payoffs in examples 2.1 and 2.2, we however obtain closed-form expressions for these tail integrals, which allows for a simple theoretical analysis.



(a) VIX **(b)** SVIX

Figure A.1: Tail behavior of VIX and SVIX (power law)

This figure shows possible option-implied moments $\mathrm{E}^{\mathbb{Q}_{q,p}}[f(R)]$ for the VIX and SVIX at different values q and p. The VIX in panel (a) uses the payoff $f(R) = \text{VIX}^2(R)$ as in example 2.1. The SVIX in panel (b) uses the payoff $f(R) = \text{SVIX}^2(R)$ as in example 2.2. Option-implied moments are computed using average S&P 500 option prices at quarterly maturity.

Example A.3 (VIX)**.** Take the payoff $\text{VIX}^2(R) = -2\log R$ as in example 2.1. The left-tail integral $\mathcal{P}^{\mathbb{Q}_{q,p}}[\text{VIX}^2]$ is finite for any $q < 0$, but infinite in the boundary case $q = 0$. Specifically, we have

$$\mathcal{P}^{\mathbb{Q}_{q,p}}[\text{VIX}^2] = \begin{cases} \infty & \text{if } q = 0 \\ -\dfrac{2}{q} P(\ell)\, \ell^{-1} & \text{if } q < 0 \end{cases} .$$

However, the right-tail integral $\mathcal{C}^{\mathbb{Q}_{q,p}}[\text{VIX}^2]$ is finite for every $p \geq 1$, given by

$$\mathcal{C}^{\mathbb{Q}_{q,p}}[\text{VIX}^2] = -\frac{2}{p} C(u)\, u^{-1} .$$

In that sense, the VIX payoff constitutes a boundary case that only leads to unbounded valuations $\mathrm{E}^{\mathbb{Q}_{q,p}}[\text{VIX}^2(R)]$ when $q = 0$. This corresponds to the limiting case $a \to 0$ for piecewise-linear specifications.

Panel (a) in figure A.1 visualizes the numerical valuations $\mathrm{E}^{\mathbb{Q}_{q,p}}[\text{VIX}^2(R)]$ at different q and p, obtained from average S&P 500 option prices. As expected from our analysis, variation in the right-tail power p has relatively little visible impact on the valuations, while an increase $q \to 0$ in the left-tail power leads to valuations that approach infinity. ■

Example A.4 (SVIX)**.** Now consider the payoff $\text{SVIX}^2(R) = (R-1)^2$ as in example 2.2. For every $q \leq 0$, the left-tail integral $\mathcal{P}^{\mathbb{Q}_{q,p}}[\text{SVIX}^2]$ is finite, given by

$$\mathcal{P}^{\mathbb{Q}_{q,p}}[\text{SVIX}^2] = \frac{2}{2-q} P(\ell)\, \ell .$$

In contrast, the right-tail integral $\mathcal{C}^{\mathbb{Q}_{q,p}}[\text{SVIX}^2]$ is finite only if $p > 2$, but infinite for each $1 \leq p \leq 2$.

Specifically, it holds that

$$\mathcal{C}^{\mathbb{Q}_{q,p}}[\mathrm{SVIX}^2] = \begin{cases} \infty & \text{if } 1 \leq p \leq 2 \\ \dfrac{2}{p-2} C(u)\, u & \text{if } p > 2 \end{cases} .$$

Therefore, unlike the VIX, the SVIX is not a boundary case. It results in unbounded valuations $\mathrm{E}^{\mathbb{Q}_{q,p}}[\mathrm{SVIX}^2(R)]$ even for p that are away from the boundary at $p = 1$, which corresponds to the limit case $b \to \infty$ for piecewise-linear specifications.

Panel (b) in figure A.1 shows valuations $\mathrm{E}^{\mathbb{Q}_{q,p}}[\mathrm{SVIX}^2(R)]$ for different q and p, computed using average S&P 500 option prices. Consistent with our analysis, the choice of left-tail power q has a comparatively small impact on valuations, whereas letting the right-tail power $p \to 2$ leads to the expected increase of valuations towards infinity. ■

B Model-free moment bounds with continuous strike sets

In this section, we rigorously develop model-free moment bounds in a setting with continuous strike sets, as in section 3. Throughout this section, we thus employ the definition

$$\mathcal{B} := \left\{ \mathbb{Q} \in \mathcal{A} : P^{\mathbb{Q}}(k) = P(k), C^{\mathbb{Q}}(k) = C(k) \text{ for all } k \in [\ell, u] \right\} . \tag{B.1}$$

Model-free bounds for general moments $\mathrm{E}^{\mathbb{Q}}[f(R)]$ associated to some moment function $f \in \mathcal{F}$ over all $\mathbb{Q} \in \mathcal{B}$ can be computed from the problem

$$\inf_{\mathbb{Q}\in\mathcal{B}} / \sup\ \mathrm{E}^{\mathbb{Q}}[f(R)] , \tag{B.2}$$

whose values are denoted by $\mathcal{L}[f]$ for the lower bound and $\mathcal{U}[f]$ for the upper bound, respectively.

Notably, problem (B.2) can be decomposed into a left-tail and right-tail problem, which may be solved separately. For this decomposition, we express $\mathbb{Q} = \mathbb{Q}^L + \mathbb{Q}^M + \mathbb{Q}^R$ as a sum of positive measures, such that $\mathbb{Q}^L \in \mathcal{M}((0, \ell])$, $\mathbb{Q}^M \in \mathcal{M}((\ell, u))$, and $\mathbb{Q}^R \in \mathcal{M}([u, \infty))$. Here, we denote by $\mathcal{M}(A)$ the set of positive (Borel) measures having support contained in some set $A \subseteq \mathcal{X}$. As a continuum of option prices is observed on $[\ell, u]$, $\mathbb{Q}^M$ is uniquely identified using $\mathbb{Q}^M(R \leq k) = \partial_{k+}C(k) - \partial_{k+}C(\ell)$ for any $k \in (\ell, u)$. In contrast, $\mathbb{Q}^L$ and $\mathbb{Q}^R$ are not identified, but constrained by the local behavior of put and call prices at the extreme strikes $\ell = K_1$ and $u = K_n$, respectively. For the left-tail and right-tail problems, we define the associated sets of measures by

$$\mathcal{B}^L := \{\mathbb{Q} \in \mathcal{M}((0, \ell]) : P^{\mathbb{Q}}(\ell) = P(\ell),\ \mathrm{E}^{\mathbb{Q}}[1] = \partial_{k+}P(\ell)\} \tag{B.3}$$

$$\mathcal{B}^R := \{\mathbb{Q} \in \mathcal{M}([u, \infty)) : C^{\mathbb{Q}}(u) = C(u),\ \mathrm{E}^{\mathbb{Q}}[1] = \partial_{k-}C(u)\} . \tag{B.4}$$

Instead of dealing with the aggregate problem (B.2), the setting of this section allows us to independently treat the left-tail and right-tail problems

$$\inf_{\mathbb{Q}\in\mathcal{B}^L} / \sup\ \mathrm{E}^{\mathbb{Q}}[f(R)] , \qquad \inf_{\mathbb{Q}\in\mathcal{B}^R} / \sup\ \mathrm{E}^{\mathbb{Q}}[f(R)] , \tag{B.5}$$

from which we obtain the lower and upper bounds $\mathcal{L}^L[f]$ and $\mathcal{U}^L[f]$ as well as $\mathcal{L}^R[f]$ and $\mathcal{U}^R[f]$. Eventually, the aggregate lower and upper bounds in problem (B.2) can be equivalently determined from the tail problems (B.5) using the identities

$$\mathcal{L}[f] = \mathcal{L}^L[f] + \mathrm{E}^{\mathbb{Q}^M}[f(R)] + \mathcal{L}^R[f] \tag{B.6}$$

$$\mathcal{U}[f] = \mathcal{U}^L[f] + \mathrm{E}^{\mathbb{Q}^M}[f(R)] + \mathcal{U}^R[f] \ . \tag{B.7}$$

In the remainder of this section, we discuss the construction of model-free tail moment bounds for several classes of moments.

B.1 Bounds on differentiable convex moments

As a first class of moments, consider those associated to functions $f \in \mathcal{F}$ that are convex and twice continuously differentiable, so that a version of the static replication formula (2.1) applies. Formally, we deal with functions in the set

$$\mathcal{F}^{\mathrm{dc}}(A) := \left\{ f \in \mathcal{F} : f \text{ is twice continuously differentiable and convex on } A \subset \mathcal{X} \right\} .$$

For any function $f \in \mathcal{F}^{\mathrm{dc}}(A)$, we thus have $f''(R) \geq 0$ for $R \in A$. Left-tail moment bounds for $f \in \mathcal{F}^{\mathrm{dc}}((0,\ell])$ and right-tail moment bounds for $f \in \mathcal{F}^{\mathrm{dc}}([u,\infty))$ can then be constructed directly from the static replication formula together with model-free bounds for option prices.

To derive model-free bounds for put and call prices $P^{\mathbb{Q}}(k) := \mathrm{E}^{\mathbb{Q}}[(k-R)^+]$ and $C^{\mathbb{Q}}(k) := \mathrm{E}^{\mathbb{Q}}[(R-k)^+]$ at a given strike k in the corresponding tail, we consider problems (B.5) using $f(R) = (k-R)^+$ and $f(R) = (R-k)^+$, respectively. For convenience, the resulting lower and upper option price bounds are denoted as

$$P^{\mathcal{L}}(k) := \mathcal{L}[(k-R)^+] \ , \quad P^{\mathcal{U}}(k) := \mathcal{U}[(k-R)^+] \ , \quad C^{\mathcal{L}}(k) := \mathcal{L}[(R-k)^+] \ , \quad C^{\mathcal{U}}(k) := \mathcal{U}[(R-k)^+] \ .$$

For both puts and calls, these option price bounds are available in closed form, as provided in the following proposition. In either case, lower and upper option price bounds correspond to the minimal and maximal feasible convex extensions of the observed price function, available in piecewise-linear form. The range of possible extensions only depends on the local behavior of option prices, captured by the prices $P(\ell)$ and $C(u)$ as well as their one-sided derivatives $\partial_{k+}P(\ell)$ and $\partial_{k-}C(u)$.

Proposition B.1 (Bounds on tail option prices)**.**

(i) The model-free bounds for put prices $P^{\mathbb{Q}}(k)$ for any $k \in (0,\ell]$ are given by:

$$P^{\mathcal{L}}(k) = \big(P(\ell) + \partial_{k+}P(\ell)(k-\ell)\big)^+ \tag{B.8}$$

$$P^{\mathcal{U}}(k) = \lim_{a\to 0}\left(\frac{k-a}{\ell-a}P(\ell)\right)^+ = \frac{k}{\ell}P(\ell) \ . \tag{B.9}$$

(ii) The model-free bounds for call prices $C^{\mathbb{Q}}(k)$ for any $k \in [u,\infty)$ are given by:

$$C^{\mathcal{L}}(k) = \big(C(u) + \partial_{k-}C(u)(k-u)\big)^+ \tag{B.10}$$

$$C^{\mathcal{U}}(k) = \lim_{b\to\infty}\left(\frac{b-k}{b-u}C(u)\right)^+ = C(u) \ . \tag{B.11}$$

Unlike the lower bounds, the upper bounds in proposition B.1 are not attained by a measure in $\mathcal{B}$, but through a limiting procedure. To formally make the connection between tail moment bounds for f and the tail option price bounds, we define for small enough $0 < a < \ell$ and large enough $b > u$

$$P_a^{\mathcal{U}}(k) := \left(\frac{k-a}{\ell-a}P(\ell)\right)^+ , \qquad C_b^{\mathcal{U}}(k) := \left(\frac{b-k}{b-u}C(u)\right)^+ ,$$

noting that $P^{\mathcal{U}}(k) = \lim_{a\to 0} P_a^{\mathcal{U}}(k)$ and $C^{\mathcal{U}}(k) = \lim_{b\to\infty} C_b^{\mathcal{U}}(k)$.

With these definitions, tail moment bounds for $f \in \mathcal{F}^{\mathrm{dc}}((0,\ell])$ and $f \in \mathcal{F}^{\mathrm{dc}}([u,\infty))$ can be given as in the following proposition. Essentially, the moment bounds result from the static replication formula when using the (limiting form of) option price bounds in proposition B.1 in the integrals and performing closed-form integration. The moment bounds depend on the local behavior of option prices at the extreme strikes ℓ and u as well as the local and limiting behavior of the function f.

Proposition B.2 (Bounds on differentiable convex moments)**.**

(i) For $f \in \mathcal{F}^{\mathrm{dc}}((0,\ell])$, the model-free left-tail moment bounds are given by:

$$\begin{aligned}\mathcal{L}^L[f] &= f(\ell)\,\partial_{k+}P(\ell) - f'(\ell)\,P(\ell) + \int_0^\ell f''(K)\,P^{\mathcal{L}}(K)\,\mathrm{d}K \\ &= \partial_{k+}P(\ell)\,f\left(\ell - \frac{P(\ell)}{\partial_{k+}P(\ell)}\right)\end{aligned} \tag{B.12}$$

$$\begin{aligned}\mathcal{U}^L[f] &= f(\ell)\,\partial_{k+}P(\ell) - f'(\ell)\,P(\ell) + \lim_{a\to 0}\int_0^\ell f''(K)\,P_a^{\mathcal{U}}(K)\,\mathrm{d}K \\ &= \left(\partial_{k+}P(\ell) - \frac{P(\ell)}{\ell}\right) f(\ell) + \frac{P(\ell)}{\ell}\,\lim_{a\to 0} f(a)\ .\end{aligned} \tag{B.13}$$

(ii) For $f \in \mathcal{F}^{\mathrm{dc}}([u,\infty))$, the model-free right-tail moment bounds are given by:

$$\begin{aligned}\mathcal{L}^R[f] &= -f(u)\,\partial_{k-}C(u) + f'(u)\,C(u) + \int_u^\infty f''(K)\,C^{\mathcal{L}}(K)\,\mathrm{d}K \\ &= -\partial_{k-}C(u)\,f\left(u - \frac{C(u)}{\partial_{k-}C(u)}\right)\end{aligned} \tag{B.14}$$

$$\begin{aligned}\mathcal{U}^R[f] &= -f(u)\,\partial_{k-}C(u) + f'(u)\,C(u) + \lim_{b\to\infty}\int_u^\infty f''(K)\,C_b^{\mathcal{U}}(K)\,\mathrm{d}K \\ &= -\partial_{k-}C(u)\,f(u) + C(u)\,\lim_{b\to\infty}\frac{f(b)}{b}\ .\end{aligned} \tag{B.15}$$

B.2 Bounds on convex moments

To generalize the results in section B.1, we drop the differentiability requirement. Nevertheless, we maintain the convexity property, which turns out to be substantially helpful for deriving model-free moment bounds. Accordingly, we now consider the set of functions

$$\mathcal{F}^{\mathrm{c}}(A) := \left\{ f \in \mathcal{F} : f \text{ is convex on } A \subset \mathcal{X} \right\} .$$

Evidently, it holds that $\mathcal{F}^{\mathrm{dc}}(A) \subset \mathcal{F}^{\mathrm{c}}(A)$.

Even without applicability of the static replication formula, we still arrive at equivalent left-tail moment bounds for $f \in \mathcal{F}^{\mathrm{c}}((0,\ell])$ and right-tail moment bounds for $f \in \mathcal{F}^{\mathrm{c}}([u,\infty))$ as in proposition B.2. The resulting closed-form expressions are again stated in the following proposition.

Proposition B.3 (Bounds on convex moments)**.**

(i) For $f \in \mathcal{F}^c((0,\ell])$, the model-free left-tail moment bounds are given by:

$$\mathcal{L}^L[f] = \partial_{k+}P(\ell)\, f\left(\ell - \frac{P(\ell)}{\partial_{k+}P(\ell)}\right) \tag{B.16}$$

$$\mathcal{U}^L[f] = \left(\partial_{k+}P(\ell) - \frac{P(\ell)}{\ell}\right) f(\ell) + \frac{P(\ell)}{\ell} \lim_{a\to 0} f(a) \ . \tag{B.17}$$

(ii) For $f \in \mathcal{F}^c([u,\infty))$, the model-free right-tail moment bounds are given by:

$$\mathcal{L}^R[f] = -\partial_{k-}C(u)\, f\left(u - \frac{C(u)}{\partial_{k-}C(u)}\right) \tag{B.18}$$

$$\mathcal{U}^R[f] = -\partial_{k-}C(u)\, f(u) + C(u) \lim_{b\to\infty} \frac{f(b)}{b} \ . \tag{B.19}$$

B.3 Bounds on general moments

Finally, we turn to the general case of arbitrary $f \in \mathcal{F}$, without further requirements as in sections B.1 and B.2. For this generalization, we exploit the availability of bounds on convex moments in section B.2 by convexifying f in a certain way. Specifically, in the respective tails, we construct the lower convex envelopes $\mathcal{E}^L[f] \in \mathcal{F}^c((0,\ell])$ and $\mathcal{E}^R[f] \in \mathcal{F}^c([u,\infty))$, which are defined as

$$\mathcal{E}^L[f](R) := \inf\left\{\frac{r''-R}{r''-r'}\, f(r') + \frac{R-r'}{r''-r'}\, f(r'') : r' \le R < r'' \le \ell\right\} \tag{B.20}$$

$$\mathcal{E}^R[f](R) := \inf\left\{\frac{r''-R}{r''-r'}\, f(r') + \frac{R-r'}{r''-r'}\, f(r'') : u \le r' \le R < r''\right\} \tag{B.21}$$

over all $r', r'' \in \mathcal{X}$. By construction, it holds that $\mathcal{E}^L[f](R) \le f(R)$ for $R \in (0,\ell]$ and $\mathcal{E}^R[f](R) \le f(R)$ for $R \in [u,\infty)$.

The following proposition formalizes that $\mathcal{L}^L[f] = \mathcal{L}^L[\mathcal{E}^L[f]]$ and $\mathcal{L}^R[f] = \mathcal{L}^R[\mathcal{E}^R[f]]$, using convexity and interpolation properties induced by the lower convex envelopes. Likewise, applying the same approach to $-f$, it further holds that $\mathcal{U}^L[f] = -\mathcal{L}^L[\mathcal{E}^L[-f]]$ and $\mathcal{L}^R[f] = -\mathcal{L}^R[\mathcal{E}^R[-f]]$. With these identities, closed-form expressions for lower and upper bounds immediately follow from proposition B.3.

Proposition B.4 (Bounds on general moments)**.**

(i) For $f \in \mathcal{F}$, the model-free left-tail moment bounds are given by:

$$\mathcal{L}^L[f] = \mathcal{L}^L[\mathcal{E}^L[f]] = \partial_{k+}P(\ell)\, \mathcal{E}^L[f]\left(\ell - \frac{P(\ell)}{\partial_{k+}P(\ell)}\right) \tag{B.22}$$

$$\mathcal{U}^L[f] = -\mathcal{L}^L[\mathcal{E}^L[-f]] = -\partial_{k+}P(\ell)\, \mathcal{E}^L[-f]\left(\ell - \frac{P(\ell)}{\partial_{k+}P(\ell)}\right) \tag{B.23}$$

(ii) For $f \in \mathcal{F}$, the model-free right-tail moment bounds are given by:

$$\mathcal{L}^R[f] = \mathcal{L}^R[\mathcal{E}^R[f]] = -\partial_{k-}C(u)\, \mathcal{E}^R[f]\left(u - \frac{C(u)}{\partial_{k-}C(u)}\right) \tag{B.24}$$

$$\mathcal{U}^R[f] = -\mathcal{L}^R[\mathcal{E}^R[-f]] = \partial_{k-}C(u)\, \mathcal{E}^R[-f]\left(u - \frac{C(u)}{\partial_{k-}C(u)}\right) \tag{B.25}$$

If $f \in \mathcal{F}^c((0,\ell])$ and $f \in \mathcal{F}^c([u,\infty))$, the moment bounds in proposition B.4 of course correspond to those in proposition B.3. In fact, we then have that

$$\begin{aligned}
\mathcal{E}^L[f](R) =& f(R) \ , \quad \mathcal{E}^L[-f](R) = -\frac{\ell - R}{\ell}\left(\lim_{a\to 0} f(a)\right) - \frac{R}{\ell} f(\ell) \ , \\
\mathcal{E}^R[f](R) =& f(R) \ , \quad \mathcal{E}^R[-f](R) = -f(u) - (R-u)\left(\lim_{b\to\infty} \frac{f(b)}{b}\right) \ .
\end{aligned}$$

Substituting these relations into the moment bound expressions in proposition B.4 recovers the moment bounds in proposition B.3.

B.4 Implementation

If f is twice continuously differentiable over the interval (ℓ, u), we may determine $\mathcal{I}[f]$ using a generalization of the classical static replication formula as

$$\begin{aligned}
\mathcal{I}[f] = & f(1)\left(1 - \partial_{k+}P(\ell) + \partial_{k-}C(u)\right) \\
& + f'(1)\left(P(\ell) - C(u) + (1-\ell)\,\partial_{k+}P(\ell) + (u-1)\,\partial_{k-}C(u)\right) \\
& + \int_\ell^1 f''(K)\,P(K)\,\mathrm{d}K + \int_1^u f''(K)\,C(K)\,\mathrm{d}K \ .
\end{aligned} \tag{B.26}$$

Without assuming differentiability of f, option prices can be used to obtain an option-implied probability density along the lines of Banz and Miller (1978) and Breeden and Litzenberger (1978), which then yields $\mathcal{I}[f]$ via direct integration.

C Data

Historical option data is obtained from OptionMetrics, available through WRDS, for the period from January 1996 to February 2023. We extract data for options written on the S&P 500 index, in particular closing bid and ask prices for put and call options, closing index levels, and interest rates. Bid and ask option prices are averaged to mid prices. Following a common procedure employed in the literature, we sample data on a weekly basis for each Wednesday, or otherwise the first trading day thereafter within the same week.

Given the nature of our methodology, we apply only a bare minimum of data filters prior to conducting our analysis. In this regard, we only drop in-the-money put and call options. The selection of out-of-the-money options is motivated by their generally higher liquidity. In addition, we require at least one trading day remaining to maturity for any observation in our sample, but otherwise keep all available option data.

Our methodology is formulated using normalized prices on a forward basis. As the observed price data is quoted on a spot basis, we transform prices to a forward basis by calculating the option-implied forward price at each option maturity. In this process, interest rates are interpolated linearly to match the observed option maturities. Raw option prices $V(K,\tau)$ at strike K and maturity τ are then normalized as $v(k,\tau) = e^{\tau\, r(\tau)} V(kF(\tau),\tau)/F(\tau)$, where $k = K/F(\tau)$ is the normalized strike (forward moneyness), $r(\tau)$ is the interpolated interest rate, and $F(\tau)$ is the corresponding forward underlying price.

To perform a more detailed analysis using representative option prices, we determine an empirical average surface. For this, we fix a maturity grid consisting of expirations at 0.5, 1, 3, 6, 9, and 12 months. In a first step, we determine the median number of put and call options at the predefined set of maturities by interpolating and averaging the observed counts for straddling option maturities over the full sample period. Given the median number of options, in a second step, we then fix a strike grid by determining interpolated quantiles of strike prices for the predefined maturities on each day in the sample, and subsequently determining the median of quantiles over the full sample period. After specifying the option maturities and strikes through this procedure, we determine the associated average mid prices. Figure C.1 illustrates the resulting average option price surface.

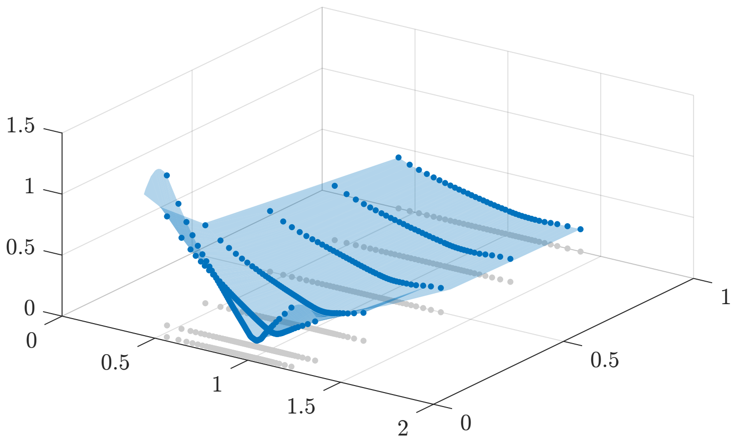


Figure C.1: Average option surface

This figure shows the empirical average option surface. The plot shows mid prices, quoted in Black-Scholes implied volatility units. The left horizontal axis shows the strike (moneyness) k; the right horizontal axis shows the time-to-maturity τ, measured in years; the vertical axis shows the implied volatility. The gray dots projected onto the horizontal plane represent the maturity-strike specifications contained in the average surface; the blue dots represent the corresponding values.