Optimal Diversification with Parameter Uncertainty*

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Abstract

Conventional investment wisdom advocates that investors should be well diversified, i.e., invest in as many assets as possible. We show instead that due to estimation errors in the inputs of portfolio strategies, the optimal level of diversification strikes a trade-off between accessing additional investment opportunities and limiting estimation risk, and thus is finite and can be relatively small. We also propose a set of procedures to select which assets are part of the restricted investment universe. Empirically, we show that limiting diversification with our method substantially outperforms unrestricted portfolios and makes portfolio theory valuable even in high-dimensional settings.

Keywords: portfolio optimization, parameter uncertainty, estimation risk.
JEL Classification: G11, G12.

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1 Introduction

This paper is in its preliminary form. The main theoretical and empirical results are already available, and we expect to have the complete discussion ready for the conference dates (June, 6-7, 2024). The empirical results we currently have largely support the theoretical intuitions, and confirm the added value of our approach.

2 Optimal diversification without parameter uncertainty

In this section, we study what is the optimal diversification, i.e., the optimal number of assets, for a mean-variance investor who knows the parameters of asset returns without uncertainty. We suppose that the investor has access to an investment universe of M assets, and needs to choose an optimal subset of $N \leq M$ assets to invest in.

Given a selected N, let $\boldsymbol{r}$ be the $N \times 1$ vector of asset excess returns, which has a mean $\boldsymbol{\mu}_N$ and a positive-definite covariance matrix $\boldsymbol{\Sigma}_N$. We denote by μ_i, σ_i^2, $\lambda_i = \mu_i/\sigma_i^2$, and $\theta_i = \mu_i/\sigma_i$ the mean return, variance, price of risk, and Sharpe ratio of asset i. A mean-variance investor with risk-aversion coefficient $\gamma > 0$ selects her portfolio $\boldsymbol{w} = (w_1, ..., w_N)'$ on the risky assets by solving the following optimization problem:

$$\boldsymbol{w}^\star = \arg\max_{\boldsymbol{w}} U(\boldsymbol{w}), \tag{1}$$

where $U(\boldsymbol{w})$ is the mean-variance utility of portfolio $\boldsymbol{w}$,

$$U(\boldsymbol{w}) = \boldsymbol{w}'\boldsymbol{\mu}_N - \frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma}_N\boldsymbol{w}. \tag{2}$$

Because we assume that $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$ are known, the solution to (1) is feasible and equal to

$$\boldsymbol{w}^\star = \frac{1}{\gamma}\boldsymbol{\Sigma}_N^{-1}\boldsymbol{\mu}_N, \tag{3}$$

which we call the *mean-variance (MV) portfolio*. The maximum utility delivered by the MV portfolio is

$$U(\boldsymbol{w}^\star) = \frac{\theta_N^2}{2\gamma} \quad \text{with} \quad \theta_N^2 = \boldsymbol{\mu}_N' \boldsymbol{\Sigma}_N^{-1} \boldsymbol{\mu}_N, \tag{4}$$

and θ_N^2 is the maximum achievable squared Sharpe ratio among the N assets.

Now, *how does $U(\boldsymbol{w}^\star)$ depend on the number of assets N?* Clearly, it has to be a non-decreasing function of N because, if adding an asset cannot improve the utility, the investor can set a weight of zero on the asset and keep the same utility level as that without the additional asset. That is, without parameter uncertainty, it is optimal to set $N = M$. To relate $U(\boldsymbol{w}^\star)$ and N more explicitly, we rely on the following assumption about the correlation matrix of asset returns, $\boldsymbol{P}_N$.

Assumption 1 *The asset returns are equicorrelated, i.e., their correlation matrix is*

$$\boldsymbol{P}_N = (1-\rho_N)\boldsymbol{I}_N + \rho_N \mathbf{1}_N \mathbf{1}_N', \tag{5}$$

where $-\frac{1}{N-1} < \rho_N < 1$ so that $\boldsymbol{P}_N$ is positive definite.

Under Assumption 1, the maximum utility $U(\boldsymbol{w}^\star)$ depends on the correlation structure of the asset returns via a single parameter, ρ_N. This assumption is also useful in a setting with parameter uncertainty because errors in correlations are known to hurt performance substantially (Ledoit and Wolf, 2003; Chung et al., 2022). The next proposition shows how $U(\boldsymbol{w}^\star)$ simplifies under Assumption 1.[1]

Proposition 1 *Let Assumption 1 hold. Then, the maximum utility $U(\boldsymbol{w}^\star) = \theta_N^2/(2\gamma)$ with*

$$\theta_N^2 = \frac{N}{1-\rho_N}\delta_N(\bar{\boldsymbol{\theta}}_N), \quad \delta_N(\bar{\boldsymbol{\theta}}_N) = \bar{\theta}_{N,2} - \frac{N\rho_N}{1-\rho_N+N\rho_N}\bar{\theta}_{N,1}^2, \tag{6}$$

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[1] The proofs of all results are available in Section C of the Supplementary Material.

where $\bar{\boldsymbol{\theta}}_N = (\bar{\theta}_{N,1}, \bar{\theta}_{N,2})$ *with*[2]

$$\bar{\theta}_{N,k} = \frac{1}{N} \sum_{i=1}^{N} \theta_i^k. \tag{7}$$

Moreover, $\lim_{N\to\infty} U(\boldsymbol{w}^\star) = +\infty$ *if and only if there is enough cross-sectional variation in the assets' Sharpe ratios in the sense that* $\lim_{N\to\infty} N(\bar{\theta}_{N,2} - \bar{\theta}_{N,1}^2) = +\infty$.[3]

Proposition 1 allows us to express the maximum utility, which is a function of $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$, only via ρ_N and $\bar{\boldsymbol{\theta}}_N$. Moreover, it shows that this maximum utility is typically unbounded with N. In Figure 1, we depict $U(\boldsymbol{w}^\star)$ as a function of N between one and M for $\rho_N = (0.2, 0.5, 0.8)$ and assets' monthly Sharpe ratios θ_i calibrated to a dataset of $M = 96$ portfolios sorted on size and book-to-market (96S-BM) spanning July 1963 to August 2023.[4] The figure shows that $U(\boldsymbol{w}^\star)$ improves with the level of diversification, as expected. Moreover, it is not a monotonic function of the correlation ρ_N.[5]

The analysis in this section shows that increasing the level of diversification always improves the performance of the MV portfolio. However, this result assumes that the investor knows the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, while these are unknown in practice. Therefore, in the next section, we study the optimal number of assets when there is parameter uncertainty.

3 Optimal diversification with parameter uncertainty

We now study the optimal level of diversification when the unknown $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$ are estimated. Specifically, given a sample of T i.i.d. asset excess returns, $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T)$, we consider

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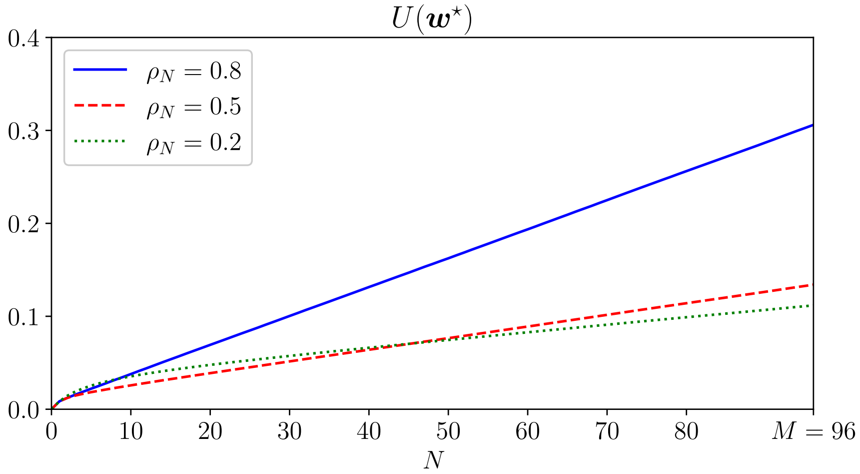
[2] It holds that $\bar{\theta}_{N,2} \geq \bar{\theta}_{N,1}^2$.

[3] When there is no cross-sectional variation in the assets' Sharpe ratios, i.e., $\theta_i = \theta_1$ for all i, we can show that $\lim_{N\to\infty} U(\boldsymbol{w}^\star) < +\infty$. Indeed, in that case we have $\bar{\theta}_{N,2} = \bar{\theta}_{N,1}^2 = \theta_1^2$, and thus $U(\boldsymbol{w}^\star) = \frac{\theta_1^2}{2\gamma}\frac{N}{1-\rho+N\rho}$ from (6), which converges to a finite value of $\theta_1^2/(2\gamma\rho)$ as $N \to \infty$.

[4] We remove four portfolios sorted on size and book-to-market for which there is missing data.

[5] We study the relation between $U(\boldsymbol{w}^\star)$ and ρ_N in Section A of the Supplementary Material.

Figure 1: Utility of the mean-variance portfolio as a function of N and ρ_N



Notes. This figure depicts $U(\boldsymbol{w}^\star)$ in Equation (6) as a function of the number of assets N under the assumption that asset returns are equicorrelated with a correlation equal to $\rho_N = (0.2, 0.5, 0.8)$. We calibrate the assets' monthly Sharpe ratios θ_i to a dataset of $M = 96$ portfolios sorted on size and book-to-market spanning July 1963 to August 2023. Starting with $N = 1$ asset chosen randomly, we compute $U(\boldsymbol{w}^\star)$. Then, we add a randomly selected asset not previously selected, and compute $U(\boldsymbol{w}^\star)$ again. We continue this procedure until $N = M$. We repeat this procedure $10,000$ times and depict the average $U(\boldsymbol{w}^\star)$ over all draws. We consider a risk-aversion coefficient $\gamma = 1$.

the following sample estimates of $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$:

$$\hat{\boldsymbol{\mu}}_N = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}_t \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_N = \frac{1}{T}\sum_{t=1}^{T} (\boldsymbol{r}_t - \hat{\boldsymbol{\mu}}_N)(\boldsymbol{r}_t - \hat{\boldsymbol{\mu}}_N)', \tag{8}$$

and we require $T > N$ so that $\hat{\boldsymbol{\Sigma}}_N$ is invertible. The resulting *sample mean-variance (SMV) portfolio* is

$$\hat{\boldsymbol{w}}^\star = \frac{1}{\gamma}\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N. \tag{9}$$

In this section, we study the optimal number of assets N for two portfolios. First, the SMV portfolio $\hat{\boldsymbol{w}}^\star$ in (9), which is not robust to parameter uncertainty. Second, the so-called two-fund rule that scales down $\hat{\boldsymbol{w}}^\star$ to optimize out-of-sample performance, as in Kan and Zhou (2007) and Kan and Lassance (2024). Our objective in this section, as well as Section 4, is only to find the optimal value of N. We treat the problem of *which* N assets out of the M available ones we should select in Section 7.3.

3.1 Optimal diversification for the SMV portfolio

As pioneered by Kan and Zhou (2007), we measure the performance of the SMV portfolio $\hat{\boldsymbol{w}}^\star$ under parameter uncertainty via its *expected out-of-sample utility* (EU):

$$EU(\hat{\boldsymbol{w}}^\star) = \mathbb{E}[U(\hat{\boldsymbol{w}}^\star)] = \mathbb{E}\Big[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}_N - \frac{\gamma}{2}\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}_N\hat{\boldsymbol{w}}^\star\Big], \tag{10}$$

where the expectation is evaluated under the true distribution of asset returns. Kan and Zhou (2007) and follow-up papers like Tu and Zhou (2011), DeMiguel et al. (2015), Kan et al. (2021), Lassance et al. (2024), and Lassance et al. (2023) all assume that asset returns are multivariate normally distributed, which is analytically useful but unrealistic in practice. Therefore, we follow Kan and Lassance (2024) who study the EU of various sample portfolio rules when asset returns are multivariate *elliptically* distributed. Like them, we use the representation of the multivariate elliptical distribution in El Karoui (2010, 2013).

Assumption 2 *Asset returns are multivariate elliptically distributed as*

$$\boldsymbol{r} \overset{d}{=} \boldsymbol{\mu}_N + (\tau_N\boldsymbol{\Sigma}_N)^{1/2}\boldsymbol{z}_N, \tag{11}$$

where $\boldsymbol{z}_N \sim \mathcal{N}(\boldsymbol{0}_N, \boldsymbol{I}_N)$*,* τ_N *is a univariate random variable fulfilling* $\mathbb{E}[\tau_N] = 1$*, and* $\boldsymbol{z}_N$ *and* τ_N *are mutually independent.*

The distribution of τ_N in Assumption 2 does not depend on $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$ and captures the effect of fat tails. For example, we recover the multivariate normal distribution when $\tau_N = 1$, and the multivariate t-distribution when $\tau_N \sim (\nu_N - 2)/\chi^2_{\nu_N}$, where $\chi^2_{\nu_N}$ denotes a chi-square distribution with $\nu_N > 2$ degrees of freedom.

In the next proposition, we present the EU of the SMV portfolio when asset returns are multivariate elliptically distributed, which will then allow us to determine the optimal diversification N for an investor holding the SMV portfolio.

Proposition 2 (Kan and Lassance (2024), Proposition 7) *Let* $T > N+4$, $\boldsymbol{M} = \boldsymbol{I}_T - \mathbf{1}_T\mathbf{1}_T'/T$, $\boldsymbol{Z}_N$ *be a* $T \times N$ *matrix of independent standard normal random variables,* $\boldsymbol{\Lambda}_N = \operatorname{diag}\big(\tau_{N,1}^{1/2}, \dots, \tau_{N,T}^{1/2}\big)$ *be a diagonal matrix of independent copies of* $\tau_N^{1/2}$*, and* $\boldsymbol{\Lambda}_N$ *and* $\boldsymbol{Z}_N$ *are mutually independent. Then, under Assumption 2, the expected out-of-sample utility of the sample mean-variance portfolio* $\hat{\boldsymbol{w}}^\star$ *in* (9) *is*

$$EU(\hat{\boldsymbol{w}}^\star) = \frac{1}{2\gamma}\frac{T}{T-N-2}\left[\left(2\kappa_{N,1} - \frac{c_N\kappa_{N,2}T}{T-N-2}\right)\theta_N^2 - \frac{c_N\kappa_{N,3}N}{T-N-2}\right], \tag{12}$$

where

$$c_N = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}, \tag{13}$$

and $\kappa_{N,1}$*,* $\kappa_{N,2}$*, and* $\kappa_{N,3}$ *are functions of only* N*,* T*, and the distribution of* τ_N*:*

$$\kappa_{N,1} = \frac{T-N-2}{N}\mathbb{E}\big[\operatorname{tr}\big((\boldsymbol{Z}_N'\boldsymbol{\Lambda}_N\boldsymbol{M}\boldsymbol{\Lambda}_N\boldsymbol{Z}_N)^{-1}\big)\big], \tag{14}$$

$$\kappa_{N,2} = \frac{(T-N-2)^2}{c_N N}\mathbb{E}\big[\operatorname{tr}\big((\boldsymbol{Z}_N'\boldsymbol{\Lambda}_N\boldsymbol{M}\boldsymbol{\Lambda}_N\boldsymbol{Z}_N)^{-2}\big)\big], \tag{15}$$

$$\kappa_{N,3} = \frac{(T-N-2)^2}{c_N NT}\mathbb{E}\big[\mathbf{1}_T'\boldsymbol{\Lambda}_N\boldsymbol{Z}_N(\boldsymbol{Z}_N'\boldsymbol{\Lambda}_N\boldsymbol{M}\boldsymbol{\Lambda}_N\boldsymbol{Z}_N)^{-2}\boldsymbol{Z}_N'\boldsymbol{\Lambda}_N\mathbf{1}_T\big], \tag{16}$$

provided the expectations exist. Moreover, when asset returns are multivariate normally distributed, $\kappa_{N,1}$*,* $\kappa_{N,2}$*, and* $\kappa_{N,3}$ *are all equal to one.*

Several comments are in order. First, the expectations involved in $\kappa_{N,1}$, $\kappa_{N,2}$, and $\kappa_{N,3}$ do not have a closed-form expression, but can be evaluated via Monte Carlo simulations given the distribution of τ_N. In Section 5, we explain how we estimate them from a sample of historical returns. As shown by Kan and Lassance (2024), they increase with estimation risk N/T and the heaviness of the tails.

Second, because parameter uncertainty in $\boldsymbol{\mu}_N$ and $\boldsymbol{\Sigma}_N$ severely impacts the EU of the SMV portfolio, as formalized in Proposition 2, we no longer expect more diversification to always be better. Instead, when selecting the optimal N that maximizes (12), there is a tradeoff to strike between improving the population performance (i.e., increasing θ_N^2) and limiting estimation risk (i.e., decreasing N/T).

Third, we cannot directly find the value of N maximizing (12), for two reasons. The first reason is that θ_N^2 and the distribution of τ_N are unknown because they are population parameters. We leave this first issue aside for now; in Section 5, we explain how we estimate the parameters on which the optimal N depends for the different portfolio rules considered in this paper. The second reason is that θ_N^2 is not an explicit function of N. Therefore, we employ Assumption 1, under which, by plugging θ_N^2 in (6) into (12), the EU of the SMV portfolio becomes

$$EU(\hat{\boldsymbol{w}}^\star) = \frac{1}{2\gamma}\frac{NT}{T-N-2}\left[\frac{\delta_N(\bar{\boldsymbol{\theta}}_N)}{1-\rho_N}\left(2\kappa_{N,1} - \frac{c_N\kappa_{N,2}T}{T-N-2}\right) - \frac{c_N\kappa_{N,3}}{T-N-2}\right] \tag{17}$$

$$=: f_{smv}(N, T, \boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N). \tag{18}$$

The function f_{smv} depends on three things. First, the sample size T, directly but also via c_N and $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3})$. Second, the number of assets N, directly but also via c_N, the function δ_N, and $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3})$. Third, *which* N assets are selected via $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N)$. Moreover, f_{smv} depends on $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N)$ only via ρ_N and $\bar{\boldsymbol{\theta}}_N$, and on τ_N only via $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3})$. We do not highlight the dependence on γ because the N optimizing (18) does not depend on γ.

Now, as explained at the beginning of this section, we want to separate the determination of the optimal N from the decision of which N assets to select. We achieve this by replacing $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N)$ in the function f_{smv} by its counterpart for the whole investment universe of M assets, i.e., $(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M)$.[6] All in all, the way we determine the optimal N for the SMV

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[6] Moreover, in practice, the correlation ρ_N, the average of assets' Sharpe ratios $\bar{\boldsymbol{\theta}}_N$, and the distribution of τ_N are unknown and must be estimated, and these estimates are more accurate as N increases. Therefore, by avoiding optimizing N based on a noisy estimate of $(\rho_N, \bar{\boldsymbol{\theta}}_N, \tau_N)$, but instead based on $(\rho_M, \bar{\boldsymbol{\theta}}_M, \tau_M)$ that can be more accurately estimated since M is typically large, we make the optimal N less affected by estimation errors.

portfolio becomes

$$N^{\star}_{smv} = \underset{N \in (1,\ldots,\min(M,T-5))}{\operatorname{argmax}} f_{smv}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M), \tag{19}$$

where we search N up to $\min(M, T-5)$ because we assume $T > N+4$ in Proposition 2.

We now illustrate the magnitude of $N^{\star}_{smv}$. We take $\rho_M = (0.2, 0.5, 0.8)$ and we calibrate $\bar{\boldsymbol{\theta}}_M$ to the 96S-BM dataset, which yields $\bar{\boldsymbol{\theta}}_M = (0.125, 0.0169)$. We consider two distributions for asset returns. First, the multivariate normal distribution, i.e., $\tau_M = 1$ and $\kappa_{N,1} = \kappa_{N,2} = \kappa_{N,3} = 1$. Second, the multivariate t-distribution, i.e., $\tau_M \sim (\nu_M - 2)/\chi^2_{\nu_M}$, with $\nu_M = 6$.

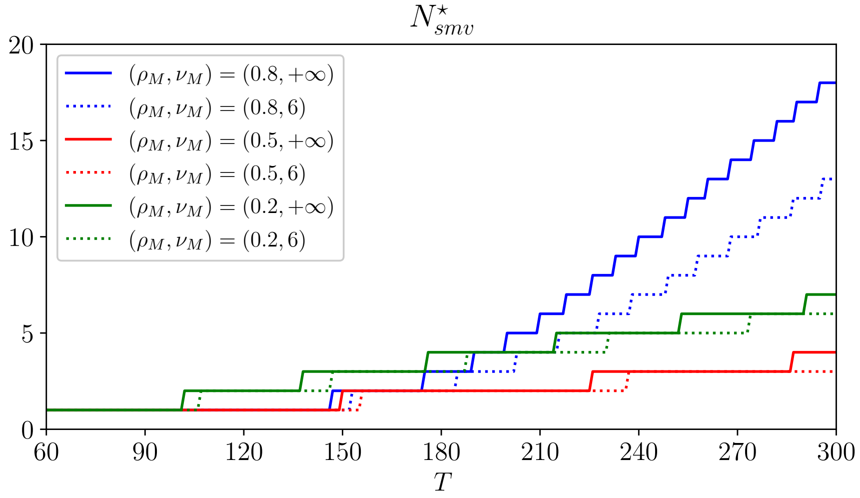
Figure 2 depicts how $N^{\star}_{smv}$ evolves as a function of the sample size T; we expect it to increase with T because, as T increases, the SMV portfolio becomes a better estimate of the MV portfolio and the optimal N for the latter is unbounded. We find that $N^{\star}_{smv}$ is small compared to T. For instance, with $T = 240$ and $\rho_M = 0.5$, we have $N^{\star}_{smv} = 3$ both for the multivariate normal and multivariate t-distributions. For a larger $\rho_M = 0.8$ we find a larger $N^{\star}_{smv}$ when T is large enough but it remains small for practical values of T. For example, consider $\rho_M = 0.8$ and sample sizes $T = 120$, 180 and 240. We have $N^{\star}_{smv} = 1$, 3, and 10 for the normal distribution, and $N^{\star}_{smv} = 1$, 2, and 7 for the t-distribution, respectively.

In summary, because the SMV portfolio weights $\hat{\boldsymbol{w}}^{\star}$ are highly affected by estimation errors, it is better to keep a small number of weights to estimate and abandon opportunities from additional assets. This is consistent with the well-known result in the literature that the SMV portfolio performs badly out of sample even for relatively small N.

3.2 Optimal diversification for the two-fund rule

The optimal number of assets for the SMV portfolio is typically small, but this is because this portfolio is not robust to parameter uncertainty and thus suffers too much from the additional estimation risk incurred when adding assets in the portfolio.

Figure 2: Optimal diversification for the sample mean-variance portfolio



Notes. This figure depicts the optimal number of assets for the sample mean-variance portfolio, $N^{\star}_{smv}$ in Equation (19), as a function of the sample size T under the assumption that asset returns are equicorrelated with a correlation equal to $\rho_M = (0.2, 0.5, 0.8)$. We calibrate $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3})$ to the multivariate normal distribution (solid lines) and to the multivariate t-distribution with $\nu_M = 6$ degrees of freedom (dotted lines). We calibrate $\bar{\boldsymbol{\theta}}_M$ to a dataset of $M = 96$ portfolios sorted on size and book-to-market spanning the period July 1963 to August 2023, which yields $\bar{\boldsymbol{\theta}}_M = (0.125, 0.0169)$.

In this section, we study instead the optimal diversification for the so-called *two-fund rule* that is more robust to parameter uncertainty because it optimally scales down the SMV portfolio so as to deliver the greatest EU possible. This two-fund rule is of the form

$$\hat{\boldsymbol{w}}(\alpha) = \alpha \hat{\boldsymbol{w}}^{\star} = \frac{\alpha}{\gamma} \hat{\boldsymbol{\Sigma}}_N^{-1} \hat{\boldsymbol{\mu}}_N, \tag{20}$$

where $\alpha \in \mathbb{R}$ is the combination coefficient. In the next proposition, we exploit Proposition 2 to derive the EU of the two-fund rule when asset returns are multivariate elliptically distributed, the resulting optimal combination coefficient α, and which EU it delivers.

Proposition 3 *Let $T > N + 4$ and Assumption 2 hold. Then, the expected out-of-sample utility of the two-fund rule $\hat{\boldsymbol{w}}(\alpha)$ in (20) is*

$$EU(\hat{\boldsymbol{w}}(\alpha)) = \frac{1}{2\gamma} \frac{T}{T - N - 2} \left[\left(2\alpha\kappa_{N,1} - \frac{c_N \alpha^2 \kappa_{N,2} T}{T - N - 2} \right) \theta_N^2 - \frac{c_N \alpha^2 \kappa_{N,3} N}{T - N - 2} \right]. \tag{21}$$

Moreover, the optimal combination coefficient $\alpha^\star$ *maximizing* (21) *is*

$$\alpha^\star = \frac{T-N-2}{c_N T}\frac{\kappa_{N,1}\theta_N^2}{\kappa_{N,2}\theta_N^2 + \kappa_{N,3}\frac{N}{T}}, \tag{22}$$

and the resulting expected out-of-sample utility is

$$EU(\hat{\boldsymbol{w}}(\alpha^\star)) = \frac{1}{2\gamma c_N}\frac{\kappa_{N,1}^2\theta_N^4}{\kappa_{N,2}\theta_N^2 + \kappa_{N,3}\frac{N}{T}} \tag{23}$$

Given that the optimal two-fund rule controls for parameter uncertainty by scaling down the SMV portfolio as a function of N, T, and the heaviness of the tails, we now expect that the optimal diversification for this portfolio rule is larger than that for the SMV portfolio.

Following a similar reasoning to that in Section 3.1, we find the optimal N by maximizing (23) after replacing θ_N^2 by that under Assumption 1, and replacing $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N)$ by $(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M)$. That is, we determine the optimal N for the two-fund rule as

$$N_{2f}^\star = \underset{N\in(1,\ldots,\min(M,T-5))}{\operatorname{argmax}} f_{2f}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M). \tag{24}$$

where

$$f_{2f}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M) = \frac{N}{2\gamma c_N(1-\rho_M)} \times \frac{\kappa_{N,1}^2\delta_N(\bar{\boldsymbol{\theta}}_M)^2}{\kappa_{N,2}\delta_N(\bar{\boldsymbol{\theta}}_M) + \kappa_{N,3}\frac{1-\rho_M}{T}} \tag{25}$$

Based on Equation (24), we can show that in the multivariate normal case where $\kappa_{N,1} = \kappa_{N,2} = \kappa_{N,3} = 1$, $N_{2f}^\star$ is below half of the sample size $T/2$ (or equal to M if M is sufficiently below $T/2$), and closer to $T/2$ the higher the value of ρ_M. To see this, consider the N that maximizes the first multiplicative term of f_{2f}, i.e., N/c_N. We have

$$\frac{N}{c_N} = \frac{N(T-N-1)(T-N-4)}{(T-2)(T-N-2)} \approx \frac{N(T-N-4)}{(T-2)}, \tag{26}$$

and the latter is maximized by

$$\underset{N\in(1,\ldots,\min(M,T-5))}{\operatorname{argmax}} \frac{N(T-N-4)}{(T-2)} = \min([T/2-2], M). \tag{27}$$

Turning to the second multiplicative term of f_{2f} in (24), it is easy to show that if $\kappa_{N,1} = \kappa_{N,2} = \kappa_{N,3} = 1$, it is a decreasing function of N and thus is maximized by $N = 1$, which moves $N^{\star}_{2f}$ below (27). However, the sensitivity of this second term to N depends on how sensitive $N\rho_M/(1 - \rho_M + N\rho_M)$ is to N, and it is less so as ρ_M increases.[7] Thus, we expect $N^{\star}_{2f}$ to stay close to (27) if ρ_M is not too small. Finally, in the multivariate elliptical case, $\kappa_{N,1}$, $\kappa_{N,2}$, and $\kappa_{N,3}$ are different from one and depend on N, but the following example and later simulations suggest that $N^{\star}_{2f}$ still remains reasonably close to that under normality.

Figure 3 illustrates the optimal diversification $N^{\star}_{2f}$ for the two-fund rule in the same setting as that in Figure 2. We find as predicted above that $N^{\star}_{2f}$ is below $T/2$ and equal to the full size of the investment universe, M, once $T/2$ is too much above M. We also see that in line with the above, $N^{\star}_{2f}$ gets closer to $T/2$ when ρ_M increases. Finally, while fat tails negatively impact the optimal diversification for the SMV portfolio in Figure 2, they now positively impact the optimal diversification for the two-fund rule, $N^{\star}_{2f}$. This can be explained because while the SMV portfolio is not robust to the impact of fat tails, the optimal two-fund rule scales down the SMV portfolio according to the heaviness of the tails via $\kappa_{N,1}$, $\kappa_{N,2}$, and $\kappa_{N,3}$. Therefore, as shown by Kan and Lassance (2024, Proposition 5), the optimal two-fund rule often performs better when asset returns are multivariate elliptically distributed instead of multivariate normally distributed, hence we find a larger $N^{\star}_{2f}$ under elliptical returns too.

4 Optimal diversification for three-fund rules

The two-fund rule in (20) controls for parameter uncertainty by optimally combining the SMV portfolio with the risk-free asset. We now study the optimal diversification for *three-fund rules* that add another portfolio rule to further alleviate parameter uncertainty and improve

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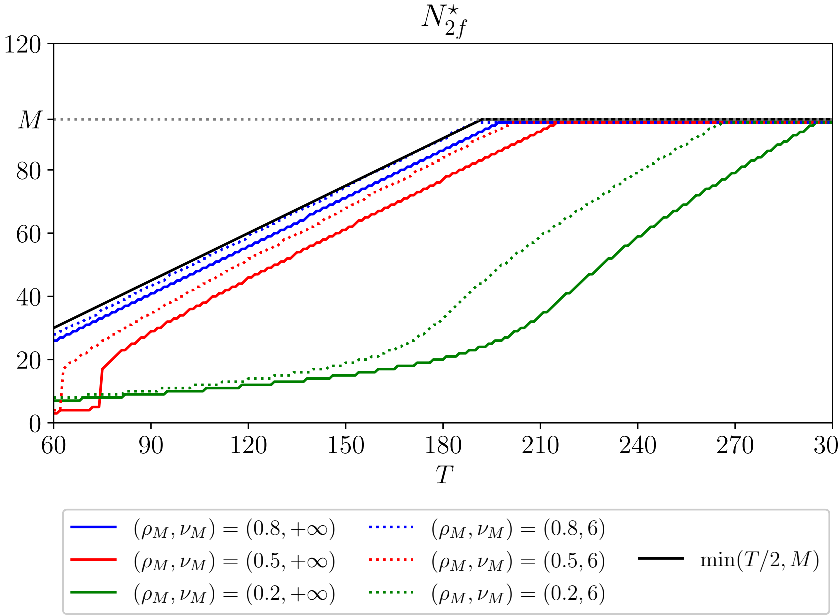
[7] Specifically, $\frac{\partial}{\partial \rho_M}\left[\frac{\partial}{\partial N}\frac{N\rho_M}{1-\rho_M+N\rho_M}\right] = \frac{1-\rho_M-N\rho_M}{(1-\rho_M+N\rho_M)^3}$ is negative if $\rho_M \geq 1/(N+1)$, which typically holds.

Figure 3: Optimal diversification for the two-fund rule



Notes. This figure depicts the optimal number of assets for two-fund rule, $N^{\star}_{2f}$ in Equation (24), as a function of the sample size T under the assumption that asset returns are equicorrelated with a correlation equal to $\rho_M = (0.2, 0.5, 0.8)$. We calibrate $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3})$ to the multivariate normal distribution (solid lines) and to the multivariate t-distribution with $\nu_M = 6$ degrees of freedom (dotted lines). We calibrate $\bar{\boldsymbol{\theta}}_M$ to a dataset of $M = 96$ portfolios sorted on size and book-to-market spanning the period July 1963 to August 2023, which yields $\bar{\boldsymbol{\theta}}_M = (0.125, 0.0169)$.

the EU. We consider two different three-fund rules: that based on the global minimum-variance (GMV) portfolio as in Kan and Zhou (2007) and that based on the equally weighted (EW) portfolio as in Tu and Zhou (2011) and Lassance et al. (2023).

4.1 Three-fund rule with the global minimum-variance portfolio

We first consider the GMV portfolio as an additional portfolio rule to invest in,

$$\boldsymbol{w}_g = \frac{\boldsymbol{\Sigma}_N^{-1}\mathbf{1}_N}{\mathbf{1}_N'\boldsymbol{\Sigma}_N^{-1}\mathbf{1}_N}, \tag{28}$$

which is more robust to parameter uncertainty than the MV portfolio because it does not depend on the mean vector $\boldsymbol{\mu}_N$. We introduce the following parameters:

$$\mu_{g,N} = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_N}{\mathbf{1}_N'\boldsymbol{\Sigma}^{-1}\mathbf{1}_N}, \quad \sigma_{g,N}^2 = \frac{1}{\mathbf{1}_N'\boldsymbol{\Sigma}^{-1}\mathbf{1}_N}, \quad \lambda_{g,N} = \frac{\mu_{g,N}}{\sigma_{g,N}^2}, \quad \theta_{g,N}^2 = \frac{\mu_{g,N}^2}{\sigma_{g,N}^2}, \tag{29}$$

which stand for the mean return, variance, price of risk, and squared Sharpe ratio of the GMV portfolio $\boldsymbol{w}_g$ computed on N assets. Moreover, we denote the difference between the maximum squared Sharpe ratio and that of the GMV portfolio as

$$\psi_{g,N}^2 = \theta_N^2 - \theta_{g,N}^2 \geq 0. \tag{30}$$

Under parameter uncertainty, we define the three-fund rule that invests in the SMV portfolio, the sample GMV (SGMV) portfolio, and the risk-free asset as

$$\hat{\boldsymbol{w}}(\boldsymbol{\alpha}) = \frac{\alpha_1}{\gamma}\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N + \frac{\alpha_2}{\gamma}\hat{\boldsymbol{\Sigma}}_N^{-1}\mathbf{1}_N, \tag{31}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is the vector of combination coefficients. We call $\hat{\boldsymbol{w}}(\boldsymbol{\alpha})$ the *GMV-three-fund rule*. In the next proposition, we exploit Proposition 2 and Kan and Lassance (2024, Proposition 8) to derive the EU of the GMV-three-fund rule in (31) when asset returns are multivariate elliptically distributed, the resulting optimal combination coefficients $\boldsymbol{\alpha}^\star$, and which EU they deliver.

Proposition 4 *Let $T > N + 4$ and Assumption 2 hold. Then, the expected out-of-sample utility of the GMV-three-fund rule $\hat{\boldsymbol{w}}(\boldsymbol{\alpha})$ in* (31) *is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\alpha})) = \frac{1}{2\gamma}\frac{T}{T-N-2}\Bigg[2\kappa_{N,1}\big(\alpha_1\theta_N^2 + \alpha_2\lambda_{g,N}\big) - \frac{c_N T}{T-N-2} \times \left(\alpha_1^2\left(\kappa_{N,2}\theta_N^2 + \kappa_{N,3}\frac{N}{T}\right) + \frac{\alpha_2^2\kappa_{N,2}}{\sigma_{g,N}^2} + 2\alpha_1\alpha_2\kappa_{N,2}\lambda_{g,N}\right)\Bigg]. \quad (32)$$

Moreover, the optimal combination coefficients $\boldsymbol{\alpha}^\star = (\alpha_1^\star, \alpha_2^\star)$ maximizing (32) *are*

$$\alpha_1^\star = \frac{T-N-2}{c_N T}\frac{\kappa_{N,1}\psi_{g,N}^2}{\kappa_{N,2}\psi_{g,N}^2 + \kappa_{N,3}\frac{N}{T}}, \quad (33)$$

$$\alpha_2^\star = \frac{T-N-2}{c_N T}\frac{\kappa_{N,3}}{\kappa_{N,2}}\frac{\kappa_{N,1}\frac{N}{T}\mu_{g,N}}{\kappa_{N,2}\psi_{g,N}^2 + \kappa_{N,3}\frac{N}{T}}. \quad (34)$$

and the resulting expected out-of-sample utility is

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\alpha}^\star)) = \frac{\kappa_{N,1}^2}{2\gamma c_N}\frac{\theta_N^2\psi_{g,N}^2 + \frac{\kappa_{N,3}}{\kappa_{N,2}}\frac{N}{T}\theta_{g,N}^2}{\kappa_{N,2}\psi_{g,N}^2 + \kappa_{N,3}\frac{N}{T}}. \quad (35)$$

Proposition 4 shows that for an investor holding the optimal GMV-three-fund rule, the optimal diversification is found by maximizing the EU in (35) with respect to N. However, this EU depends on θ_N^2 and $\theta_{g,N}^2$, which are not explicit functions of N. Therefore, as in Sections 2 and 3, we assume that asset returns are equicorrelated, i.e., Assumption 1, which allows us to express θ_N^2 and $\theta_{g,N}^2$ as more explicit functions of N. Proposition 1 shows what θ_N^2 becomes under Assumption 1, and we derive $\theta_{g,N}^2$ in the next proposition.

Proposition 5 *Let Assumption 1 hold. Then, the squared Sharpe ratio of the global minimum-variance portfolio is*[8]

$$\theta_{g,N}^2 = \frac{N}{1-\rho_N} r_{g,N}(\bar{\theta}_{N,1}, \bar{\lambda}_N, \bar{\boldsymbol{\sigma}}_N), \quad r_{g,N}(\bar{\theta}_{N,1}, \bar{\lambda}_N, \bar{\boldsymbol{\sigma}}_N) = \frac{\left(\bar{\lambda}_N - \frac{N\rho_N}{1-\rho_N+N\rho_N}\bar{\theta}_{N,1}\bar{\sigma}_{N,-1}\right)^2}{\bar{\sigma}_{N,-2} - \frac{N\rho_N}{1-\rho_N+N\rho_N}\bar{\sigma}_{N,-1}^2}, \quad (36)$$

---

[8] Note that $\lim_{N\to\infty}\theta_{g,N}^2 = +\infty$ as long as $\lim_{N\to\infty} N r_{g,N}(\bar{\theta}_{N,1}, \bar{\lambda}_N, \bar{\boldsymbol{\sigma}}_N) = +\infty$, which is because the GMV portfolio can attain a zero variance as the number of assets grows indefinitely.

where $\bar{\theta}_{N,1}$ *is defined in* (7), $\bar{\boldsymbol{\sigma}}_N = (\bar{\sigma}_{N,-1}, \bar{\sigma}_{N,-2})$, *and*

$$\bar{\lambda}_N = \frac{1}{N}\sum_{i=1}^{N} \lambda_i, \quad \bar{\sigma}_{N,k} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^k. \tag{37}$$

Following a similar reasoning to that in Section 3, we can now find the optimal N for the GMV-three-fund rule by maximizing (35) after replacing (θ^2, θ_g^2) by that under Assumption 1, and replacing $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N)$ by $(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M)$. That is, we determine the optimal N for the GMV-three-fund rule as

$$N_{3f,g}^{\star} = \underset{N \in (1,\ldots,\min(M,T-5))}{\text{argmax}} \; f_g(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M), \tag{38}$$

where

$$\begin{aligned} &f_{3f,g}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M) \\ &= \frac{N\kappa_{N,1}^2}{2\gamma c_N(1-\rho_M)} \times \frac{\delta_N(\bar{\boldsymbol{\theta}}_M)\big(\delta_N(\bar{\boldsymbol{\theta}}_M) - r_{g,N}(\bar{\theta}_{M,1}, \bar{\lambda}_M, \bar{\boldsymbol{\sigma}}_M)\big) + \frac{\kappa_{N,3}}{\kappa_{N,2}}\frac{1-\rho_M}{T} r_{g,N}(\bar{\theta}_{M,1}, \bar{\lambda}_M, \bar{\boldsymbol{\sigma}}_M)}{\kappa_{N,2}\big(\delta_N(\bar{\boldsymbol{\theta}}_M) - r_{g,N}(\bar{\theta}_{M,1}, \bar{\lambda}_M, \bar{\boldsymbol{\sigma}}_M)\big) + \kappa_{N,3}\frac{1-\rho_M}{T}}. \end{aligned} \tag{39}$$

Note that as for the two-fund rule, the objective function $f_{3f,g}$ is proportional to N/c_N, which as shown in (26)–(27) is maximized by N slightly below $T/2$. The simulations in Section 6 confirm that $N_{3f,g}^{\star}$ hovers around $T/2$. Therefore, allowing for an investment in the SGMV portfolio does not change much the optimal diversification.[9]

4.2 Three-fund rule with the equally weighted portfolio

We now consider the EW portfolio as an additional portfolio rule to invest in,

$$\boldsymbol{w}_{ew} = \frac{\mathbf{1}_N}{N}, \tag{40}$$

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[9] That the EU of both the optimal two-fund rule and the GMV-three-fund rule is proportional to N/c_N can be explained as follows: the proportionality to $1/c_N$ is because both the SMV portfolio and the SGMV portfolio are proportional to $\hat{\boldsymbol{\Sigma}}_N^{-1}$, and the proportionality to N is because both the squared Sharpe ratio of the MV portfolio (θ_N^2) and the GMV portfolio ($\theta_{g,N}^2$) are proportional to N under Assumption 1.

which is not subject to parameter uncertainty. We introduce the following parameters:

$$\mu_{ew,N} = \boldsymbol{w}_{ew}'\boldsymbol{\mu}_N, \quad \sigma_{ew,N}^2 = \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}_N\boldsymbol{w}_{ew}, \quad \lambda_{ew,N} = \frac{\mu_{ew,N}}{\sigma_{ew,N}^2}, \quad \theta_{ew,N}^2 = \frac{\mu_{ew,N}^2}{\sigma_{ew,N}^2}, \tag{41}$$

which stand for the mean return, variance, price of risk, and squared Sharpe ratio of the EW portfolio $\boldsymbol{w}_{ew}$ computed on N assets. Moreover, we denote the difference between the maximum squared Sharpe ratio and that of the EW portfolio as

$$\psi_{ew,N}^2 = \theta_N^2 - \theta_{ew,N}^2 \geq 0. \tag{42}$$

Under parameter uncertainty, we define the three-fund rule that invests in the SMV portfolio, the EW portfolio, and the risk-free asset as

$$\hat{\boldsymbol{w}}(\boldsymbol{\beta}) = \frac{\beta_1}{\gamma}\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N + \frac{\beta_2}{\gamma}\boldsymbol{w}_{ew}, \tag{43}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$ is the vector of combination coefficients. We call $\hat{\boldsymbol{w}}(\boldsymbol{\beta})$ the *EW-three-fund rule*. In the next proposition, we exploit Proposition 2 to derive the EU of the EW-three-fund rule (31) when asset returns are multivariate elliptically distributed, the resulting optimal combination coefficients $\boldsymbol{\alpha}^\star$, and which EU they deliver.[10] While Tu and Zhou (2011) constrain $\beta_1 + \beta_2 = 1$ in their setting, we follow Lassance et al. (2023) and do not impose this constraint.

Proposition 6 *Let $T > N + 4$ and Assumption 2 hold. Then, the expected out-of-sample utility of the EW-three-fund rule $\hat{\boldsymbol{w}}(\boldsymbol{\beta})$ in* (43) *is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\beta})) = \frac{1}{2\gamma}\times \left[\frac{2\beta_1\kappa_{N,1}\theta_N^2 T}{T-N-2} + 2\beta_2\mu_{ew,N} - \frac{\beta_1^2 c_N T^2}{(T-N-2)^2}\left(\kappa_{N,2}\theta_N^2 + \kappa_{N,3}\frac{N}{T}\right) - \beta_2^2\sigma_{ew,N}^2 - \frac{2\beta_1\beta_2\kappa_{N,1}\mu_{ew,N}T}{T-N-2}\right]. \tag{44}$$

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[10] Proposition 6 extends the asymptotic result in Lassance et al. (2023, Proposition 7) to finite samples.

Moreover, the optimal combination coefficients $\boldsymbol{\beta}^{\star} = (\beta_1^{\star}, \beta_2^{\star})$ *maximizing* (44) *are*

$$\beta_1^{\star} = \frac{T-N-2}{T} \frac{\kappa_{N,1}\psi_{ew,N}^2}{\kappa_{N,1}^2\psi_{ew,N}^2 + d_N}, \tag{45}$$

$$\beta_2^{\star} = \frac{d_N \lambda_{ew,N}}{\kappa_{N,1}^2\psi_{ew,N}^2 + d_N}, \tag{46}$$

where

$$d_N = c_N \kappa_{N,3} \frac{N}{T} + (c_N \kappa_{N,2} - \kappa_{N,1}^2)\theta_N^2, \tag{47}$$

and the resulting expected out-of-sample utility is

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\beta}^{\star})) = \frac{1}{2\gamma} \frac{\kappa_{N,1}^2 \theta_N^2 \psi_{ew,N}^2 + d_N \theta_{ew,N}^2}{\kappa_{N,1}^2\psi_{ew,N}^2 + d_N}. \tag{48}$$

The EU of the optimal EW-three-fund rule depends on θ_N^2 and $\theta_{ew,N}^2$, which are not explicit functions of N. Therefore, as usual we assume that asset returns are equicorrelated, i.e., Assumption 1, which allows us to express θ_N^2 and $\theta_{ew,N}^2$ as more explicit functions of N. Proposition 1 deals with θ_N^2 and we derive $\theta_{ew,N}^2$ in the next proposition.

Proposition 7 *Let Assumption 1 hold. Then, the squared Sharpe ratio of the equally weighted portfolio is*[11]

$$\theta_{ew,N}^2 = \frac{N}{1-\rho_N} r_{ew,N}(\bar{\mu}_N, \bar{\sigma}_{N,2}, \overline{\mathrm{cov}}_N), \quad r_{ew,N}(\bar{\mu}_N, \bar{\sigma}_{N,2}, \overline{\mathrm{cov}}_N) = \frac{(1-\rho_N)(\bar{\mu}_N)^2}{\bar{\sigma}_{N,2} + (N-1)\overline{\mathrm{cov}}_N}, \tag{49}$$

where $\bar{\sigma}_{N,2}$ *is defined in* (37) *and*

$$\bar{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} \mu_i, \quad \overline{\mathrm{cov}}_N = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \rho \sigma_i \sigma_j. \tag{50}$$

As usual, we now find the optimal N for the EW-three-fund rule by maximizing (48) after replacing $(\theta_N^2, \theta_{ew,N}^2)$ by that under Assumption 1, and replacing $(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N, \tau_N)$ by

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[11] Unlike θ_N^2 and $\theta_{g,N}^2$ that typically diverge to infinity with N under Assumption 1, $\lim_{N\to\infty} \theta_{ew,N}^2 = (\bar{\mu}_\infty)^2/\overline{\mathrm{cov}}_\infty$, which is typically finite.

$(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M)$. That is, we determine the optimal N for the EW-three-fund rule as

$$N^{\star}_{3f,ew} = \underset{N \in (1,\ldots,\min(M,T-5))}{\operatorname{argmax}} f_{3f,ew}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M), \tag{51}$$

where

$$f_{3f,ew}(N, T, \boldsymbol{\mu}_M, \boldsymbol{\Sigma}_M, \tau_M) = \frac{N}{2\gamma(1-\rho_M)} \times$$
$$\frac{\kappa_{N,1}^2 \delta_N(\bar{\boldsymbol{\theta}}_M)\Big(\delta_N(\bar{\boldsymbol{\theta}}_M) - r_{ew,N}(\bar{\mu}_M, \bar{\sigma}_{M,2}, \overline{\text{cov}}_M)\Big) + \Big(\frac{(1-\rho_M)c_N\kappa_{N,3}}{T} + (c_N\kappa_{N,2} - \kappa_{N,1}^2)\delta_N(\bar{\boldsymbol{\theta}}_M)\Big) r_{ew,N}(\bar{\mu}_M, \bar{\sigma}_{M,2}, \overline{\text{co}}}{\kappa_{N,1}^2\Big(\delta_N(\bar{\boldsymbol{\theta}}_M) - r_{ew,N}(\bar{\mu}_M, \bar{\sigma}_{M,2}, \overline{\text{cov}}_M)\Big) + \Big(\frac{(1-\rho_M)c_N\kappa_{N,3}}{T} + (c_N\kappa_{N,2} - \kappa_{N,1}^2)\delta_N(\bar{\boldsymbol{\theta}}_M)\Big)} \tag{52}$$

Although it does not appear as clearly as for the two-fund rule and the GMV-three-fund rule via N/c_N as a proportionality factor in the objective function, the simulations in Section 6 also show that $N^{\star}_{3f,ew}$ hovers around $T/2$.

5 Estimation of parameters

In this section, we explain how we estimate the different parameters that are needed as inputs in our different portfolio rules to determine the optimal number of assets (i.e., $N^{\star}_{smv}$, $N^{\star}_{2f}$, $N^{\star}_{3f,g}$, $N^{\star}_{3f,ew}$) and the optimal combination coefficients (i.e., $\alpha^{\star}$, $\alpha^{\star}_1$, $\alpha^{\star}_2$, $\beta^{\star}_1$, $\beta^{\star}_2$).

5.1 Estimation of elliptical fat tails

The first set of parameters are those that determine the impact of the fat tails of the elliptical distribution, i.e., $\kappa_{N,1}$, $\kappa_{N,2}$, and $\kappa_{N,3}$ in (14)–(16). We compute these parameters based on τ_M to find the optimal number of assets and based on τ_N to find the optimal combination coefficients; we focus on τ_N here as the two cases are similar. To speed up the computation, we use the high-dimensional approximation of these parameters. Specifically, from El Karoui (2010, 2013), we have that as $N, T \to \infty$ and $N/T \to \phi \in (0,1)$, $(\kappa_{N,1}, \kappa_{N,2}, \kappa_{N,3}) \to$

$(\tilde{\kappa}_{N,1}, \tilde{\kappa}_{N,2}, \tilde{\kappa}_{N,1})$, where $\tilde{\kappa}_{N,1} \geq 1$ is the unique positive solution to

$$\mathbb{E}\big[(1-\phi+\phi\tilde{\kappa}_{N,1}\tau_N)^{-1}\big] = 1, \tag{53}$$

and $\tilde{\kappa}_{N,2} \geq \tilde{\kappa}_{N,1}^2$ is given by

$$\tilde{\kappa}_{N,2} = (1-\phi)\left(\tilde{\kappa}_{N,1}^{-2} - \mathbb{E}\left[\frac{\phi\tau_N^2}{(1-\phi+\phi\tilde{\kappa}_{N,1}\tau_N)^2}\right]\right)^{-1}. \tag{54}$$

Kan and Lassance (2024) show that this high-dimensional approximation is accurate for typical values of N and T.

Given this result, we need to estimate $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$. We estimate them in two different ways as in Kan and Lassance (2024). First, we assume that the asset returns follow a multivariate t-distribution, i.e., $\tau_N \sim (\nu_N - 2)/\chi^2_{\nu_N}$, and we estimate the number of degrees of freedom ν_N by maximum likelihood from a sample of T historical returns $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T)$, giving us $\hat{\nu}_N$. Then, we can use the closed-form expression for $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$ when asset returns are multivariate t-distributed in Kan and Lassance (2024, Proposition 6), which yields that the estimate of $\tilde{\kappa}_{N,1}$, denoted $\tilde{\kappa}^{\nu}_{N,1}$, is the unique positive solution to

$$ye^y E_{\hat{\nu}_M/2}(y) = \phi_N \quad \text{with} \quad y = \frac{(\hat{\nu}_N - 2)\phi_N\tilde{\kappa}^{\nu}_{N,1}}{2(1-\phi_N)} \quad \text{and} \quad \phi_N = \frac{N}{T}, \tag{55}$$

where $E_n(x) = \int_1^\infty t^{-n}e^{-xt}\mathrm{d}t$ is the exponential integral, and the estimate of $\tilde{\kappa}_{N,2}$ is

$$\tilde{\kappa}^{\nu}_{N,2} = \frac{2(\tilde{\kappa}^{\nu}_{N,1})^2(1-\phi_N)}{\hat{\nu}_N - \tilde{\kappa}^{\nu}_{N,1}(\hat{\nu}_N - 2)}. \tag{56}$$

The second method we use to estimate $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$ does not specify a particular parametric distribution for τ_N, but instead, relies on the following sample estimate of the distribution of τ_N proposed by El Karoui (2010, 2013):

$$\hat{\tau}_{N,t} = \frac{(\boldsymbol{r}_t - \hat{\boldsymbol{\mu}}_N)'(\boldsymbol{r}_t - \hat{\boldsymbol{\mu}}_N)}{\frac{1}{T}\sum_{i=1}^T(\boldsymbol{r}_i - \hat{\boldsymbol{\mu}}_N)'(\boldsymbol{r}_i - \hat{\boldsymbol{\mu}}_N)}, \quad t = 1, \ldots, T, \tag{57}$$

which is consistent as $N \to \infty$. Using $\hat{\tau}_{N,t}$, we estimate $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$ with their sample counterparts, i.e., the estimate of $\tilde{\kappa}_{N,1}$, denoted $\tilde{\kappa}^s_{N,1}$, is the unique positive solution to

$$\frac{1}{T}\sum_{t=1}^{T}\big(1-\phi_N+\phi_N\tilde{\kappa}^s_{N,1}\hat{\tau}_{N,t}\big)^{-1}=1, \tag{58}$$

and the estimate of $\tilde{\kappa}_{N,2}$ is

$$\tilde{\kappa}^s_{N,2}=(1-\phi_N)\left((\tilde{\kappa}^s_{N,1})^{-2}-\frac{1}{T}\sum_{t=1}^{T}\frac{\phi_N\hat{\tau}^2_{N,t}}{(1-\phi_N+\phi_N\tilde{\kappa}^s_{N,1}\hat{\tau}_{N,t})^2}\right)^{-1}. \tag{59}$$

5.2 Estimation of portfolios' performance

The second set of parameters are those that determine the performance of the MV, GMV, and EW portfolios and on which the optimal combination coefficients depend: θ^2_N in (4), $\psi^2_{g,N}$ in (30), $\psi^2_{ew,N}$ in (42), $\mu_{g,N}$ in (29), and $\lambda_{ew,N}$ in (41).

For $\mu_{g,N}$ and $\lambda_{ew,N}$, we rely on the estimates that are unbiased when asset returns are multivariate normally distributed[12], i.e.,

$$\hat{\mu}_{g,N}=\frac{\mathbf{1}_N'\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N}{\mathbf{1}_N'\hat{\boldsymbol{\Sigma}}_N^{-1}\mathbf{1}_N}, \tag{60}$$

$$\hat{\lambda}_{ew,N}=\frac{T-3}{T}\frac{\boldsymbol{w}_{ew}'\hat{\boldsymbol{\mu}}_N}{\boldsymbol{w}_{ew}'\hat{\boldsymbol{\Sigma}}_N\boldsymbol{w}_{ew}}. \tag{61}$$

For θ^2_N, $\psi^2_{g,N}$, and $\psi^2_{ew,N}$, the sample estimates, obtained by plugging in $(\hat{\boldsymbol{\mu}}_N,\hat{\boldsymbol{\Sigma}}_N)$, are severely biased. Therefore, we estimate them using the adjusted estimates in Kan and Zhou (2007) and Kan and Wang (2023) that correct the unbiased estimates to ensure they are positive. Specifically, let $\hat{\theta}^2_N=\hat{\boldsymbol{\mu}}_N'\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N$, $\hat{\psi}^2_{g,N}=\hat{\theta}^2_N-\hat{\theta}^2_{g,N}$, and $\hat{\psi}^2_{ew,N}=\hat{\theta}^2_N-\hat{\theta}^2_{ew,N}$ be the sample estimates of θ^2_N, $\psi^2_{g,N}$, and $\psi^2_{ew,N}$, where $\hat{\theta}^2_{g,N}=(\mathbf{1}_N'\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N)^2/(\mathbf{1}_N'\hat{\boldsymbol{\Sigma}}_N^{-1}\mathbf{1}_N)$ and

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[12] We can show that when asset returns are multivariate elliptically distributed as in Assumption 2, $\hat{\mu}_{g,N}$ and $\hat{\lambda}_{ew,N}$ are also unbiased in the high-dimensional asymptotic regime as $N,T\to\infty$ and $N/T\to\phi\in(0,1)$.

$\hat{\theta}^2_{ew,N} = (\boldsymbol{w}'_{ew}\hat{\boldsymbol{\mu}}_N)^2/(\boldsymbol{w}'_{ew}\hat{\boldsymbol{\Sigma}}\boldsymbol{w}_{ew})$. Then, the adjusted estimates are

$$\hat{\theta}^2_{N,a} = \frac{(T-N-2)\hat{\theta}^2_N - N}{T} + \frac{2(\hat{\theta}^2_N)^{\frac{N}{2}}(1+\hat{\theta}^2_N)^{\frac{2-T}{2}}}{T \times B_{\hat{\theta}^2_N/(1+\hat{\theta}^2_N)}\left(\frac{N}{2}, \frac{T-N}{2}\right)}, \tag{62}$$

$$\hat{\psi}^2_{g,N,a} = \frac{(T-N-1)\hat{\psi}^2_{g,N} - (N-1)}{T} + \frac{2(\hat{\psi}^2_{g,N})^{\frac{N-1}{2}}(1+\hat{\psi}^2_{g,N})^{\frac{2-T}{2}}}{T \times B_{\hat{\psi}^2_{g,N}/(1+\hat{\psi}^2_{g,N})}\left(\frac{N-1}{2}, \frac{T-N+1}{2}\right)}, \tag{63}$$

$$\hat{\psi}^2_{ew,N,a} = \frac{(T-N-2)\hat{\psi}^2_{ew,N} - (N-1)(1+\hat{\theta}^2_{ew,N})}{T} + \frac{2(1+\hat{\theta}^2_{ew,N})^{\frac{T-N}{2}}(\hat{\psi}^2_{ew,N})^{\frac{N-1}{2}}(1+\hat{\theta}^2_N)^{\frac{3-T}{2}}}{T \times B_{\hat{\psi}^2_{ew,N}/(1+\hat{\theta}^2_N)}\left(\frac{N-1}{2}, \frac{T-N}{2}\right)}, \tag{64}$$

where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t$ is the incomplete beta function.

5.3 Estimation of assets' correlation and marginal performance

The third and last set of parameters are those that control the dependence between the assets, i.e., ρ_M under Assumption 1, and the marginal performance of the assets, i.e., $(\bar{\theta}_{M,1}, \bar{\theta}_{M,2})$ in (7), $(\bar{\lambda}_M, \bar{\sigma}_{M,-1}, \bar{\sigma}_{M,-2})$ in (37), and $(\bar{\mu}_M, \bar{\sigma}_{M,2}, \overline{\mathrm{cov}}_M)$ in (50). A first way to proceed would be to estimate them separately to control for their bias. However, we need to estimate them together in a coherent way to ensure that $\delta_N(\bar{\boldsymbol{\theta}}_M) \geq 0$ in (6), $r_{g,N}(\bar{\theta}_{N,1}, \bar{\lambda}_N, \bar{\boldsymbol{\sigma}}_N) \geq 0$ in (36), $r_{ew,N}(\bar{\mu}_N, \bar{\sigma}_{N,2}, \overline{\mathrm{cov}}_N) \geq 0$ in (49), and also $\delta_N(\bar{\boldsymbol{\theta}}_M) \geq r_{g,N}(\bar{\theta}_{N,1}, \bar{\lambda}_N, \bar{\boldsymbol{\sigma}}_N)$ and $\delta_N(\bar{\boldsymbol{\theta}}_M) \geq r_{ew,N}(\bar{\mu}_N, \bar{\sigma}_{N,2}, \overline{\mathrm{cov}}_N)$. Therefore, we simply estimate them by plugging in the sample estimates of $\boldsymbol{\mu}_M$ and $\boldsymbol{\Sigma}_M$, i.e., $\hat{\boldsymbol{\mu}}_M$ and $\hat{\boldsymbol{\Sigma}}_M$ in (8). This simple estimation method is reasonable because, as we show in the simulation analysis of Section 6, the estimated optimal values of N have a small sensitivity to these parameters.

6 Simulation analysis

In this section, we run simulations with multivariate elliptically distributed data, as in Assumption 2, to analyze two things. First, the magnitude and variability of estimated optimal values of N for the SMV portfolio and the two-fund and three-fund rules. Second, whether our optimal values of N deliver close to an optimal performance even when they are subject to estimation errors and that the underlying assumption that asset returns are equicorrelated, Assumption 1, is not fulfilled.

6.1 Estimated optimal levels of diversification

We now analyze the estimated optimal diversification for the SMV portfolio ($\hat{N}^{\star}_{smv}$), the two-fund rule ($\hat{N}^{\star}_{2f}$), the GMV-three-fund rule ($\hat{N}^{\star}_{3f,g}$), and the EW-three-fund rule ($\hat{N}^{\star}_{3f,ew}$). These are the estimated counterparts of $N^{\star}_{smv}$ in (19), $N^{\star}_{2f}$ in (24), $N^{\star}_{3f,g}$ in (38), and $N^{\star}_{3f,ew}$ in (51) following the estimation methodology presented in Section 5.

We calibrate $\boldsymbol{\mu}_M$ and $\boldsymbol{\Sigma}_M$ to the 96S-BM dataset considered previously (i.e., $M = 96$), and draw $K = 10{,}000$ times $T = 120$ monthly excess returns from a multivariate t-distribution with a number of degrees of freedom $\nu_M = (6, 8, \infty)$, where $\nu_M = \infty$ corresponds to the multivariate normal distribution.[13] Figure 4 depicts boxplots of the estimated optimal N for the different values of T and ν across the K simulations. We make four main observations. First, in line with Figure 2, the optimal N for the SMV portfolio is very small, because it is not robust to parameter uncertainty. Second, the optimal N for the two-fund and three-fund rules is slightly below $T/2$, which is expected to be the case when ρ_M is close to one as explained in Section 3.2 and observed in Figure 3.[14] Third, the lower the value of ν, and thus the fatter the tails, the higher the optimal N, even though it stays around $T/2$. This is also consistent with Section 3.2 and Figure 3. Finally, the boxplots for the two-fund and three-fund rules are remarkably stable around $T/2$, meaning that our optimal values of N are not much affected by estimation errors in the underlying parameters.

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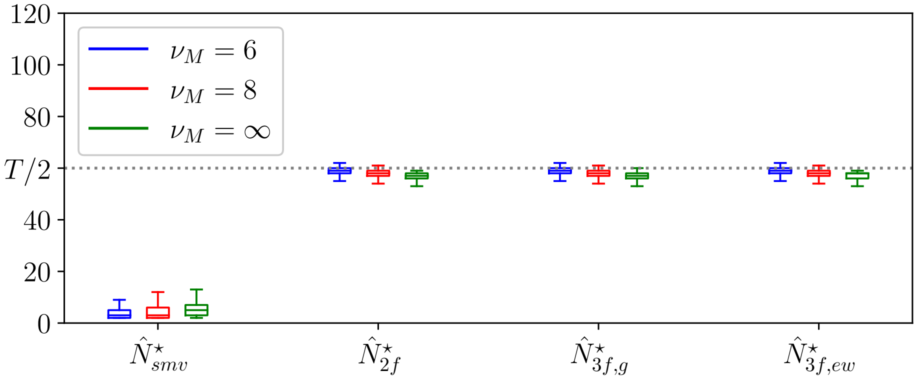
[13] Given that we simulate returns from a multivariate t-distribution, we rely on the estimates $\tilde{\kappa}^{\nu}_{N,1}$ and $\tilde{\kappa}^{\nu}_{N,2}$ of the parameters $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$ on which the optimal values of N depend. We refer to Section 5.1 for details about the definition and role of these two parameters.

[14] For the 96S-BM dataset, we have $\rho_M = 0.74$.

Figure 4: Estimated optimal levels of diversification



Notes. This figure depicts boxplots of the estimated optimal number of assets that maximizes the expected out-of-sample utility of the SMV portfolio ($\hat{N}^\star_{smv}$), two-fund rule ($\hat{N}^\star_{2f}$), GMV-three-fund rule ($\hat{N}^\star_{3f,g}$), and EW-three-fund rule ($\hat{N}^\star_{3f,ew}$). These are the estimated counterparts of $N^\star_{smv}$ in (19), $N^\star_{2f}$ in (24), $N^\star_{3f,g}$ in (38), and $N^\star_{3f,ew}$ in (51) following the estimation methodology presented in Section 5. We calibrate the mean and covariance matrix of asset returns to a dataset of $M = 96$ portfolios sorted on size and book-to-market spanning the period July 1963 to August 2023. The boxplots are obtained by simulating 10,000 times a sample of $T = 120$ multivariate t-distributed returns with $\nu_M = (6, 8, \infty)$ degrees of freedom.

6.2 Optimality of our levels of diversification

We now use simulations to investigate how optimal are our decision rules for selecting the optimal number of assets N when asset returns are indeed multivariate elliptically distributed as in Assumption 2 but 1) asset returns violate Assumption 1 according to which all correlations are equal, 2) there are estimation errors in the optimal N, and 3) the choice of which N assets to keep out of the M available ones is decided randomly. The main conclusion of the analysis below is that our selection of the optimal N delivers an EU close to the maximum even in this situation. In the empirical analysis of Section 7, we propose different selection rules to improve the performance relative to a random selection of the N assets.

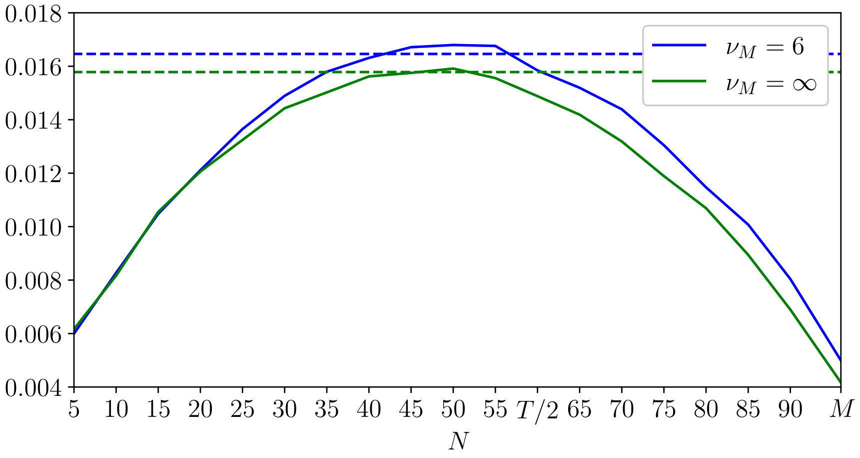
We conduct this analysis as follows. We calibrate $\boldsymbol{\mu}_M$ and $\boldsymbol{\Sigma}_M$ to the 96S-BM dataset, meaning that $\boldsymbol{\Sigma}_M$ violates Assumption 1 about equicorrelation, and draw $K = 10{,}000$ times L monthly excess returns from a multivariate t-distribution with a number of degrees of freedom $\nu_M = (6, \infty)$, where $\nu_M = \infty$ corresponds to the multivariate normal distribution. In each of the K simulations, we use these L returns to implement a rolling window exercise similar to that in the empirical analysis of Section 7. Specifically, given a sample size T and number of assets N, we randomly select N assets out of the M available ones, estimate the portfolio of these N assets on the last T months, and evaluate the out-of-sample portfolio return on the next month. We then roll the window by one month and proceed similarly until we have used the whole sample of L returns. This procedure gives us a time series of out-of-sample portfolio returns $r_{t,k}$, where $t = 1, \ldots, L - T$ and $k = 1, \ldots, K$, from which we compute the expected out-of-sample utility as

$$EU = \frac{1}{K} \sum_{k=1}^{K} \Big(\hat{\mu}_k - \frac{\gamma}{2} \hat{\sigma}_k^2 \Big) \quad \text{with} \quad \hat{\mu}_k = \frac{1}{L-T} \sum_{t=1}^{L-T} r_{t,k}, \quad \hat{\sigma}_k^2 = \frac{1}{L-T} \sum_{t=1}^{L-T} (r_{t,k} - \hat{\mu}_k)^2. \tag{65}$$

We implement this exercise with a total number of returns of $L = 300$, a sample size of $T = 120$, a risk-aversion coefficient of $\gamma = 1$, and either a fixed number of assets that ranges from $N = 5$ to $N = M = 96$ with a step size of five (six for the last step) or our estimated optimal N that varies across the K simulations. We only consider the two-fund portfolio rule in Section 3.2; the results are similar for the three-fund rules in Section 4.

We report the results of this experiment in Figure 5. We observe that our estimated optimal diversification for the two-fund rule, $\hat{N}^{\star}_{2f}$, delivers an EU that is close to the optimal one even though, unlike in our theory, it is subject to estimation errors and the equicorrelation assumption under which it is derived is violated. We can also see that for both $\nu_M = 6$ and $\nu_M = \infty$, the EU is maximized when N is slightly below $T/2$, in line with Section 3.2. Finally, reducing the number of assets from M to $N < M$ helps improve out-of-sample performance by reducing estimation risk, even though the N selected assets are chosen randomly. We expect that additional gains in out-of-sample performance can be obtained by appropriate asset selection rules, which is what we implement in the empirical analysis of Section 7.

Figure 5: Expected out-of-sample utility of the two-fund rule in simulated data



Notes. This figure depicts the expected out-of-sample utility of the two-fund rule for different number of assets N across 10,000 simulated samples of multivariate t-distributed returns with degrees of freedom $\nu_M = 6$ (solid blue), and $\nu_M = \infty$ (solid green). We calibrate the mean and covariance matrix of asset returns to a dataset of $M = 96$ portfolios sorted on size and book-to-market spanning the period July 1963 to August 2023. Each simulated sample contains 300 monthly returns, which we use to generate a time series of out-of-sample portfolio returns using rolling windows of size $T = 120$. The number of assets N varies from 5 to M. The dashed horizontal lines depict the expected out-of-sample utility when using the estimated optimal N for the two-fund rule, i.e., the estimate of $N^{\star}_{2f}$ in (24) according to the estimation method in Section 5. Which N assets out of the M available ones are selected in each rolling window is decided randomly. We use a risk-aversion coefficient of $\gamma = 1$.

7 Empirical Analysis

We now test the performance of our optimally diversified mean-variance portfolios using empirical data. In Section 7.1, we present the datasets considered. In Section 7.2, we explain the portfolio strategies whose performance will be tested. In Section 7.3, we propose different methods to select in *which* assets to invest once we obtain the optimal number of assets. In Section 7.4, we detail the out-of-sample methodology and performance measures. Finally, we report and discuss the results in Section 7.5.

7.1 Datasets

The cross-section of equities is composed of several thousands of stocks, which a challenging high-dimensional environment. To explain the cross-section more parsimoniously, the standard practice in the asset pricing literature is to group stocks into characteristic portfolios according to firm characteristics that have been shown to explain average returns. Similarly, another standard practice is to group stocks into a smaller number of industry portfolios. The portfolio problem is then to optimally allocate across the different characteristic or industry portfolios, in which the underlying stocks are simply value-weighted.

Therefore, our empirical analysis builds on six datasets of monthly excess returns of characteristic and industry portfolios, which we list in Table 1 along with the time period considered. Given that the objective of our proposed framework is to optimally reduce the number of assets in which to invest, the full investment universe must not be too small in the first place, and thus we consider datasets with M being around 100 assets.

The first dataset, 96S-BM[15], is composed of decile portfolios sorted on size and book-to-market. The second dataset, 100S-OP, is composed of decile portfolios sorted on size and operating profitability. The third dataset, 107IN-CHA, is composed of 47 industry portfolios,[16] of quintile portfolios sorted on size and book-to-market and on operating profitability and investment, and of decile portfolios sorted on momentum. These first three datasets are downloaded from Kenneth French's website. The fourth dataset, 108CHA, is composed of the long and short legs obtained from the 54 characteristics in Lassance and Martín-Utrera (2023).[17] The fifth dataset, 94IN-NV, is composed of 48 industry portfolios and of the long and short legs obtained from the 23 characteristics in Novy-Marx and Velikov (2015), which are available on Robert Novy-Marx's website. The sixth dataset, 98IN-CHA-NV, is composed of 47 industry portfolios, of quintile portfolios sorted on size and book-to-market, of decile portfolios sorted on momentum, and of the long and short legs obtained from the eight low-turnover characteristics in Novy-Marx and Velikov (2015).

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[15] We remove four assets for which there is missing data over the time period considered.

[16] We consider only 47 industry portfolios instead of 48 because there is missing data on the healthcare industry portfolio up to June 1969.

[17] We thank the authors for sharing this dataset with us.

Table 1: List of datasets considered in the empirical analysis

Dataset	M	Time period	Abbreviation
96 portfolios sorted on size and book-to-market	96	07/1963–08/2023	96S-BM
100 portfolios sorted on size and operating profitability	100	07/1963–08/2023	100S-OP
47 industry portfolios, 25 portfolios sorted on size and book-to-market, 25 portfolios sorted on operating profitability and investment, 10 portfolios sorted on momentum	107	07/1963–12/2022	107IN-CHA
108 characteristic portfolios built on long and short legs of 54 characteristics in Lassance and Martín-Utrera (2023)	108	09/1966–12/2020	108CHA
48 industry portfolios and 46 characteristic portfolios built on long and short legs of 23 characteristics in Novy-Marx and Velikov (2015)	94	07/1973–12/2013	94IN-NV
47 industry portfolios, 25 portfolios sorted on operating profitability and investment, 10 portfolios sorted on momentum, 16 characteristic portfolios built on long and short legs of eight low-turnover characteristics in Novy-Marx and Velikov (2015)	98	07/1963–12/2013	98IN-CHA-NV

Notes. This table lists the datasets of monthly excess returns we consider in the empirical analysis of Section 7. The columns report, in order, the dataset, the number of assets M, the time period, and the dataset abbreviation. The industry portfolios and the portfolios sorted on size, book-to-market, operating profitability, investment, and momentum are downloaded from Kenneth French's website. The characteristic portfolios in Novy-Marx and Velikov (2015) are downloaded from Robert Novy-Marx's website. In the construction of these datasets, we have removed assets with missing data over the time period considered. The 108CHA dataset is the same as that used by Lassance and Martín-Utrera (2023). In the construction of the 108CHA dataset, Lassance and Martín-Utrera (2023) drop characteristics with more than five percent of missing observations for more than five percent of firms with CRSP returns available for the entire sample.

Name	Funds	Description
2F	tan-rf	Combination of the sample tangent portfolio and risk-free asset. See Proposition 3.
3FG	tan-gmv-rf	Combination of the sample tangent portfolio, GMV portfolio, and risk-free asset. See Proposition 4.
3FEW	tan-ew-rf	Combination of the sample tangent portfolio, EW portfolio, and risk-free asset. See Proposition 6.
SMV	tan-rf	Sample mean-variance portfolio as defined in (9).
EW	ew	Equally-weighted portfolio as defined in (40).
GMV	gmv	Global Minimum Variance portfolio as defined in (28).
EWRF	ew-rf	Combination of the EW portfolio and risk-free asset.
GMVRF	gmv-rf	Combination of the GMV portfolio and risk-free asset.

Notes. This table lists the portfolio strategies we consider in the empirical analysis of Section 7, which are estimated with two estimates of the covariance matrix: the sample estimate in (8), and the linear shrinkage estiamte of Ledoit and Wolf (2004)

Table 2: List of portfolio strategies considered in the empirical analysis

7.2 Portfolio strategies

We evaluate the out-of-sample performance of 8 portfolio strategies listed in Table 2. The first three ones are the optimal combination rules which we study in this paper: the two-fund rule (2F), the GMV-three-fund-rule (3FG) and the EW-three-fund-rule (3FEW). The three next ones are the sample mean-variance portfolio (SMV), the equally-weighted portfolio (EW) and the global minimum variance portfolio (GMV). We also consider two other combination rules as benchmarks, the combination of the EW portfolio with the risk-free asset (EWRF) defined as $(\lambda_{ew,N}/\gamma)\boldsymbol{w}_{ew}$, and the combination of the GMV portfolio with the risk-free asset (GMVRF) defined as $\mu_{g,N}/(c_N\gamma)\boldsymbol{\Sigma}^{-1}\mathbf{1}$.

7.3 Asset Selection

Given the optimal portfolio size N, we must define a *selection rule* to decide in which N assets to invest. There is no specific optimal selection rule in the sense that it would theoretically maximize expected out-of-sample performance, therefore we propose several ad-hoc, simple

Abbr.	Name	Description
All	All assets	No selection, all assets are included in the portfolio.
MaxSR	Maximum Sharpe Ratio	Select N assets with the highest Sharpe ratios.
MinSR	Minimum Sharpe Ratio	Select N assets with the lowest Sharpe ratios.
BWSR	Best-Worst Sharpe Ratio	Select $N/2 + N \bmod 2$ assets with the highest Sharpe ratios, and $N/2$ assets with the lowest Sharpe ratios.
MinVar	Minimum Variance	Select N assets with the lowest variances.
MaxVar	Maximum Variance	Select N assets with the highest variances.
BWVar	Best-Worst Variance	Select $N/2$ assets with the highest variances, and $N/2 + N \bmod 2$ assets with the lowest variances.
PC1Low	Low Correlation to PC1	Select the N assets with the lowest correlation with the first principal component (PC1).
PC1High	High Correlation to PC1	Select the N assets with the highest correlation with the first principal component (PC1).
EPLow	Low absolute weights	Select the N assets with the lowest absolute weights.
EPHigh	High absolute weights	Select the N assets with the highest absolute weights.

Notes. This table lists the asset selection rules we use in the empirical analysis of Section 7.

Table 3: List of selection procedures considered in the empirical analysis.

and a priori sensible rules and compare their empirical performance. We do not argue that one of these selection rules is the most effective, as it is possible to define an endless number of selection rules, one of which may outperform all selection rules listed below. Rather, we show that even with very simple selection rules, restricted portfolios are able to systematically outperform unrestricted portfolios, and by a substantial margin. The selection rules that will be considered in the empirical analysis are summarized in Table 3. The first selection rule, **All assets (All)**, consists in not restricting the portfolio size and including all M assets.

Maximum Sharpe ratio (MaxSR). We can select the N assets with the highest in-sample Sharpe ratios, as we can naively expect these assets to deliver high Sharpe ratios out-of-sample. This is intuitive, but we can anticipate two issues related to leverage. Firstly, since selected assets would all exhibit favorable Sharpe ratios, we might not have access to worse performing assets which could be shorted to allocate more to high-performing ones. Secondly, we may end up shorting the ”wrong” assets, i.e. assets identified as worst because of the selection rule, even though able to actually deliver good performance.

Minimum Sharpe ratio (MinSR). Similarly to MaxSR, we can select the N assets with the lowest in-sample Sharpe ratios as we expect these assets to deliver low Sharpe ratios out-of-sample. The objective is then to profit off of these assets using short positions. We expect MinSR to be exposed to similar leverage related issues than MaxSR: not having access to better performing assets to form long positions, and having long positions on the "wrong" assets, i.e. assets identified as best because of the selection rule, even though not good performers overall.

Best-Worst Sharpe ratio (BWSR). We can select the $N/2 + N \bmod 2$ assets with the highest in-sample Sharpe ratios, and the $N/2$ assets with the lowest in-sample Sharpe ratios, where mod denotes the modulo operation.[18] BWSR addresses both leverage related issues of MaxSR and MinSR as we have access to the assets with the highest and lowest Sharpe ratios. Intuitively, we expect the portfolio to be formed of a long leg composed of assets with high in-sample Sharpe ratios, and a short leg composed of assets with low in-sample Sharpe ratios.

Since both selection rules are based on the in-sample Sharpe ratios, they depend directly on the estimator of the mean return vector $\hat{\boldsymbol{\mu}}_M$, which is a known source of significant estimation errors. We can thus expect two issues with MaxSR, MinSR and BWSR. First, selected assets may not perform as expected because high (respectively low) in-sample Sharpe ratios are not necessarily indicators of high (respectively low) out-of-sample Sharpe ratios. Second, estimates of the mean return vector are unstable, therefore the ranking of assets according to their estimated Sharpe ratio may also be unstable. Thus, the set of selected assets may change substantially from one period to another, which will lead to substantial turnover and transaction costs. BWSR is particularly exposed to the latter issue, as changes in selected assets may come from both assets with high and low in-sample Sharpe ratios. For these reasons, we also consider three selection rules based on variance, which is more a

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[18] If N is odd, we select an additional asset with a high Sharpe ratio.

stable estimator less exposed to estimation errors.

Minimum Variance (MinVar). First, we can select the N assets with the lowest in-sample variances, as we can naively expect these assets to deliver low variances out-of-sample. This selection rule is also motivated by the out-of-sample outperformance of the GMV portfolio compared to the sample mean-variance portfolio (DeMiguel et al., 2009; Jagannathan and Ma, 2003). Second, we can select the N assets with the highest in-sample variances, as we can naively expect these assets to deliver higher returns, given their higher level of risk. This is the **Maximum Variance (MaxVar)** selection rule. Last, both MinVar and MaxVar selection rules may be exposed to similar leverage related issues than MaxSR and MinSR. Therefore, we introduce the **Best-Worst Variance (BWVar)** selection rule, under which we select the $N/2 + N \bmod 2$ assets with the lowest variances[19], and the $N/2$ assets with the highest variances.

The variance-related selection rules are focused on risk, but only consider assets' idiosyncratic risk measured by variance, and fail to include another important component of risk which is correlation to the market factor. To do so, we introduce selection rules based on principal component analysis. Let us define $\boldsymbol{R}_{T,M} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M)'$ as the $T \times M$ matrix containing standardized historical returns. The sample covariance matrix of $\boldsymbol{R}_{T,M}$, $\hat{\boldsymbol{\Sigma}}_M$ as defined in (8), can be diagonalized as $\hat{\boldsymbol{\Sigma}}_M = \boldsymbol{V}\boldsymbol{L}\boldsymbol{V}'$, where $\boldsymbol{L} := \operatorname{diag}(l_1, \ldots, l_M)$ is a diagonal matrix that contains the eigenvalues which are all strictly positive, and $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M)$ is such that its columns are the eigenvectors of $\hat{\boldsymbol{\Sigma}}_M$. The weights of the first eigenvector $\boldsymbol{v}_1$, associated with the largest eigenvalue l_1, represent the market factor, whose T historical returns are given by $\boldsymbol{r}_{ma} = \boldsymbol{R}_{T,M}\boldsymbol{v}_1$. We can then define the $M \times 1$ vector $\boldsymbol{c}_M = (\operatorname{Corr}(\boldsymbol{r}_1, \boldsymbol{r}_{ma}), \ldots, \operatorname{Corr}(\boldsymbol{r}_M, \boldsymbol{r}_{ma}))$ which contains the historical correlation between the returns of each asset, and the returns of the market factor. Two selection rules can be defined based on $\boldsymbol{c}_M$: **Low Correlation to PC1 (PC1Low)**, and **High Correlation to PC1 (PC1High)** which respectively select

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[19] If N is odd, we select an additional asset with a low variance.

assets with the lowest and highest correlation to the market factor.[20]

We implement these rules as follows. First, we estimate the optimal portfolio size N following the methodology detailed in Section 5. We do not specify a particular parametric distribution for τ_M, i.e. we rely on (57)–(58)–(59) to estimate $\tilde{\kappa}_{N,1}$ and $\tilde{\kappa}_{N,2}$. Second, we follow the selection rule and identify the N selected assets. Third, we implement the portfolio strategy on the N selected assets. We follow again Section 5 to estimate optimal combination coefficients, and implement portfolio strategies with two estimates of the covariance matrix, the sample estimate in (8) and the linear shrinkage estimate of Ledoit and Wolf (2004).

All selection rules introduced above are *ex-ante* selection rules in the sense that the N assets are selected *before* computing portfolio weights. However, in-sample optimal portfolio weights can be considered by investors as a valuable source of information, which is not used with ex-ante selection rules. We can therefore compute portfolio weights on all assets first, and then select N assets based on the M obtained portfolio weights. We refer to this kind of selection rule as *ex-post*. We introduce two ex-post selection rules, **Low absolute weights (EPLow)** and **High absolute weights (EPHigh)**, which respectively select the N assets with the lowest and highest absolute weights. The intuition behind EPHigh is to select assets with extreme portfolio weights, signifying significant opportunities as these assets are either heavily shorted or bought, and eliminate spurious (i.e. close to zero) portfolio weights. The intuition behind EPLow is rather to exclude assets with extreme portfolio weights, as such extremes may indicate poorly estimated weights and associated parameters.[21]

Both of these selection rules are implemented similarly to the ex-ante rules, with the difference that portfolio weights are computed twice: once for all M assets to apply the selection rule, and then once again on the N selected assets.[22] We can anticipate a common

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[20] In unreported results, we implement a similar selection rule based on the RV2 coefficient in Smilde et al. (2008) to find the assets most correlated to the market factor. We find that results are similar.

[21] We do not report the performance of the equally-weighted portfolio with either EPLow or EPHigh, since by definition all M portfolio weights are equal.

[22] In unreported results, we show that selection rules consisting in keeping the initial portfolio weights

issue for these two ex-post rules. Assets may not be selected accurately, as the portfolio weights used to select them are estimated under very high estimation risk M/T, which can lead to extreme/inaccurate weights, and therefore asset selection.

Lastly, we report the performance of two alternative portfolio strategies that also reduce portfolio size, but differ from the selection rules listed in Table 3, in the sense that the latter rules rules all restrict portfolio size from M to an optimal portfolio size N which is known and estimated *a priori*. Some investors may want to restrict portfolio size, yet avoid constraining it to a specific size N whose computation is exposed to estimation errors. We define two selection rules within this approach.

L1 Norm Penalization (L1Pen). We solve the traditional mean-variance problem subject to a constraint that the L1norm of portfolio weights, defined as $||\boldsymbol{w}||_1 = \sum_{i=1}^{M} |w_i|$, must be smaller or equal to a tuning parameter $\delta > 0$,

$$\hat{\boldsymbol{w}}_{L1} = \underset{\boldsymbol{w}}{\operatorname{argmax}}\, \boldsymbol{w}'\hat{\boldsymbol{\mu}}_N - \frac{\gamma}{2}\boldsymbol{w}'\hat{\boldsymbol{\Sigma}}_N\boldsymbol{w} \quad \text{such that} \quad ||\boldsymbol{w}||_1 \leq \delta. \tag{66}$$

We calibrate δ using k-fold cross-validation with out-of-sample utility as decision criterion. We search in a sequence of 1,000 equally-spaced possible values of δ ranging from 0 to δ^{max}, defined as the norm of the unconstrained solution to (66),[23]

$$\delta^{max} = \left\| \frac{1}{\gamma}\hat{\boldsymbol{\Sigma}}_N^{-1}\hat{\boldsymbol{\mu}}_N \right\|_1. \tag{67}$$

DeMiguel et al. (2009) show that constraining the L1norm of portfolio weights (i) produces size-restricted portfolios, and (ii) limits the total amount of shortselling in the portfolio, which can be beneficial for out-of-sample performance (Jagannathan and Ma, 2003).[24]

Ex-Post Threshold on Weights (EPTW). Investors may want to select only the

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computed on the M assets yield very poor performances. This can be expected as these weights are computed under very high estimation risk.

[23] Any value of δ above δ^{max} would yield make the constraint in (66) non-binding.

[24] This selection rule is only applicable for the sample mean-variance portfolio.

N assets whose absolute portfolio weights are above a certain threshold w_δ, with a similar intuition to EPHigh. To avoid setting an arbitrary threshold, we calibrate w_δ by k-fold cross-validation with out-of-sample utility as decision criterion. We first define w_δ^{max} as the maximum absolute weight of the sample mean-variance portfolio,

$$w_\delta^{max} = \max \left| \frac{1}{\gamma} \hat{\boldsymbol{\Sigma}}_N^{-1} \hat{\boldsymbol{\mu}}_N \right|, \tag{68}$$

and search in a sequence of 10 equally-spaced possible values of w_δ ranging from 0 (no asset is selected) to w_δ^{max} (all assets are selected).

7.4 Performance measure

We measure portfolio performance using rolling windows. At the end of month t, we estimate portfolio k using the T previous months, and we compute its out-of-sample return in month $t+1$. We consider $T = 120$ months. We repeat this iteratively, giving a time series of $\tau - T$ out-of-sample gross returns $r_{gross,k,t}$, where τ is the total number of months in the dataset. We then compute the net out-of-sample returns, $r_{net,k,t} = r_{gross,k,t}$ if $t = T + 1$ and

$$r_{net,k,t} = (1 + r_{gross,k,t})(1 - p \times \text{turnover}_{k,t-1}) - 1 \quad \text{if} \quad t = T+2, \ldots, \tau, \tag{69}$$

where p is the proportional cost required to rebalance the portfolio and

$$\text{turnover}_{k,t} = \sum_{i=1}^{N} |w_{i,k,t} - w_{i,k,(t-1)^+}|, \quad t = T+1, \ldots, \tau, \tag{70}$$

with $w_{i,k,t}$ the weight of asset i in month t and $w_{i,k,(t-1)^+}$ the prior-month weight before rebalancing in month t. We set $p = 10$ basis points as in Ao et al. (2019). Finally, we compare the portfolio strategies in terms of *annualized out-of-sample utility net of transaction costs*,

$$U_k = 12 \times \left(\hat{\mu}_k - \frac{\gamma}{2} \hat{\sigma}_k^2 \right), \tag{71}$$

where $\hat{\mu}_k$ and $\hat{\sigma}_k^2$ are the sample mean and variance of $r_{net,k,t}$. We set $\gamma = 1$, knowing the intuition of the results remain similar with different values of γ

7.5 Results

8 Conclusion

Table 4: Annualized net out-of-sample utility with $\gamma = 1$ ($T = 120$)

		Sample $\hat{\boldsymbol{\Sigma}}_N$						Linear Shrinkage $\hat{\boldsymbol{\Sigma}}_N$					
Strat	S. Rule	96S-BM	108CHA	100S-OP	107IN-CHA	94IN-NV	98IN-CHA-NV	96S-BM	108CHA	100S-OP	107IN-CHA	94IN-NV	98IN-CHA-NV
2F	All	-0.31	-59.6	-15.3	-18.8	99.4	56.8	5.11	1.20	1.09	1.09	41.6	17.7
2F	MaxSR	-3.40	-29.7	-6.28	-24.5	7.49	-35.9	0.54	20.0	-1.17	-0.04	20.9	7.42
2F	MinSR	18.8	26.9	9.14	19.4	118	101	16.0	51.1	5.97	12.2	96.9	68.6
2F	BWSR	-4.25	16.9	-11.5	-9.41	156	173	20.6	62.7	6.53	20.6	140	91.4
2F	MinVar	1.51	-15.5	-12.5	-4.73	209	198	5.26	33.1	-2.16	5.16	135	79.5
2F	MaxVar	22.5	113	13.9	26.0	123	75.2	24.1	70.1	11.8	20.4	127	78.1
2F	BWVar	15.8	77.1	-3.55	-5.83	185	141	20.4	63.8	8.43	9.29	128	85.0
2F	PC1Low	21.8	96.0	-3.69	17.5	78.8	84.4	22.3	67.6	4.08	21.8	107	47.6
2F	PC1High	11.7	15.3	8.97	0.59	176	223	16.8	35.8	8.64	9.07	120	92.1
2F	EPLow	-2.50	-150	-6.17	-28.3	39.9	12.3	0.37	17.3	-0.24	0.44	41.0	6.78
2F	EPHigh	-348	-791	-383	-307	-731	-559	32.7	80.6	5.53	21.9	202	130
3FEW	All	1.83	-62.8	-10.8	-16.8	102	60.8	7.21	-2.56	3.01	2.38	40.3	18.5
3FEW	MaxSR	-0.48	-28.9	-2.13	-24.0	10.1	-28.7	3.61	20.3	2.17	-0.49	20.6	10.2
3FEW	MinSR	19.1	24.7	9.90	19.0	116	104	16.2	48.8	6.19	11.6	94.6	68.5
3FEW	BWSR	-5.13	17.6	-8.09	-6.46	154	173	20.2	61.5	7.83	21.1	138	91.2
3FEW	MinVar	5.73	-14.8	-6.18	-3.51	212	204	9.23	33.4	2.42	8.19	136	82.2
3FEW	MaxVar	22.8	112	16.4	27.2	123	77.5	24.4	68.2	12.7	20.2	125	77.6
3FEW	BWVar	17.7	77.5	-2.17	-1.94	194	150	20.8	62.3	8.68	9.83	130	85.9
3FEW	PC1Low	22.6	96.6	-2.24	18.6	85.5	92.6	23.0	66.2	4.05	21.8	110	50.0
3FEW	PC1High	12.0	15.1	13.0	1.34	172	232	18.0	34.6	12.4	10.5	117	92.4
3FEW	EPLow	1.84	-151	-3.76	-26.3	41.9	13.1	4.70	23.4	3.33	5.78	45.0	10.7
3FEW	EPHigh	-337	-791	-337	-275	-711	-557	24.1	48.8	-1.07	0.31	208	133
3FG	All	0.47	-60.5	-12.5	-19.7	99.1	55.4	5.77	1.21	1.79	1.31	41.9	17.9
3FG	MaxSR	-5.78	-30.2	-5.32	-19.6	13.6	-34.5	2.29	20.1	1.23	4.17	22.0	7.88
3FG	MinSR	22.6	26.1	12.4	25.4	115	103	18.9	51.2	8.01	17.0	96.8	70.4
3FG	BWSR	-0.29	16.3	-7.89	-6.92	154	173	23.1	62.9	10.1	22.5	140	91.6
3FG	MinVar	1.78	-13.9	-16.5	-3.32	209	197	6.82	33.4	-0.94	7.09	136	80.2
3FG	MaxVar	25.4	112	18.7	27.9	129	73.3	25.8	70.4	13.7	22.6	129	78.1
3FG	BWVar	23.3	77.0	4.28	2.63	186	157	24.1	64.0	12.8	14.5	129	87.0
3FG	PC1Low	24.6	100	6.70	26.8	81.8	82.1	25.7	67.9	9.51	26.6	108	48.2
3FG	PC1High	12.9	16.6	15.5	1.60	177	223	19.1	36.2	13.4	10.9	121	93.3
3FG	EPLow	-3.33	-152	-7.44	-24.6	43.5	9.14	4.04	17.7	-0.10	1.74	40.4	10.4
3FG	EPHigh	-341	-780	-324	-284	-707	-532	36.1	80.4	16.0	24.9	204	130
EW	All	8.11	2.80	8.00	7.13	3.98	5.61	8.11	2.80	8.00	7.13	3.98	5.61
EW	MaxSR	8.80	4.98	8.68	7.48	6.22	6.48	8.80	4.98	8.68	7.48	6.22	6.48
EW	MinSR	7.20	0.61	7.42	6.44	2.46	4.94	7.20	0.61	7.42	6.44	2.46	4.94
EW	BWSR	7.96	2.37	7.56	7.14	2.73	5.21	7.96	2.37	7.56	7.14	2.73	5.21
EW	MinVar	8.54	4.25	8.22	7.46	4.42	5.95	8.54	4.25	8.22	7.46	4.42	5.95
EW	MaxVar	7.94	1.32	7.90	6.75	3.53	5.22	7.94	1.32	7.90	6.75	3.53	5.22
EW	BWVar	7.56	2.30	7.46	6.83	4.07	5.46	7.56	2.30	7.46	6.83	4.07	5.46
EW	PC1Low	7.90	2.46	7.24	6.87	5.09	6.18	7.90	2.46	7.24	6.87	5.09	6.18
EW	PC1High	8.29	2.98	8.81	7.28	2.15	5.03	8.29	2.98	8.81	7.28	2.15	5.03
EWRF	All	1.42	-4.35	1.26	0.80	-3.06	-0.29	1.42	-4.35	1.26	0.80	-3.06	-0.29
GMVRF	All	1.81	-26.3	-0.98	-12.3	-11.4	-5.34	4.01	0.00	1.47	0.66	0.14	0.49
GMV	All	1.76	-49.7	4.25	-2.28	-0.34	-4.67	10.3	2.33	7.75	8.48	4.02	6.65
SMV	L1Pen	15.8	-108	11.7	-2.91	-579	0.88	23.9	67.8	5.13	10.4	-195	16.4
SMV	EPTW	-35.3	-302	-16.8	-25.9	-105	-143	-29.9	-52.1	-17.8	-33.6	-376	-349

Notes. This table reports the annualized net out-of-sample utility in percentage points for the portfolio combinations in Table 2 and the selection rules in Table 3 for the datasets in Table 1 when using the sample covariance matrix. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points, following Ao et al. (2019).

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