Untangling Illiquidity: Optimal Asset Allocation with Private Assets

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Abstract

This paper addresses the asset allocation problem of long-term investors with exposures to illiquid private asset classes (hedge funds, private equity, real estate, infrastructure, etc.). A dynamic portfolio choice model captures the temporal nature of illiquidity, where trading uncertainty hinders the investor from freely adjusting the allocation to the strategic targets. Calibrating the optimization model to analyst-based market expectations, and weighing up the risks of illiquidity against the premia and diversification potential associated with private asset classes, I quantify the welfare impact of illiquidity in investors' asset mix.

JEL codes: G11,G12
Keywords: asset allocation, (il)liquidity, portfolio choice, dynamic optimization, long-term investing

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1 Introduction

Institutional investors, such as pension funds, sovereign wealth funds, and endowments have endorsed private (or also alternative) asset classes in their portfolios in search for higher yields and diversification.[1] Long-term investors are typically well-poised to exploit the illiquidity premia that these asset classes may offer. Yet, there is still very little guidance on how the illiquidity risk of private asset classes can be incorporated into the portfolio construction process.[2]

This paper examines the problem of solving for the optimal strategic asset allocation (SAA) when investors have access to private asset classes as part of the investment mix. The liquidity friction that I consider appears in the form of trading uncertainty in a dynamic portfolio choice setting. I illustrate how the lack of predictable liquid trading opportunities affects optimal consumption and asset allocation. For a reasonable calibration of the model, I find that adding a private asset class to the strategic allocation improves risk-free equivalent consumption of the investor, even after accounting for the illiquidity risk of the investments. The improvement is economically significant.

The contribution that this paper makes is to examine the effect of illiquidity risk for private assets in the context of SAA. For this purpose, I extend a model developed by Ang et al. (2014) and provide a fast and tractable numerical algorithm to solve the underlying dynamic optimization problem. The model is calibrated to capital market assumptions used in practice by long-term investors in their SAA process (JP Morgan, 2022).

When investors are aware that relocation between some investments cannot be achieved continuously, this will inevitably affect their *ex-ante* optimal allocation decision. In anticipation of the lack of liquidity, the tilts in advance the strategic asset weights away from the illiquid investment and ensures that sufficient cash buffers are in place. Also, *ex-post* in periods without liquidity, the agent adjusts tactically the allocation to the liquid classes.

The decision process of long-term institutional investors is approximated by an infinite horizon dynamic allocation problem with intermediate consumption. The nature of liabilities for many institutional investors, such as spending rules or continuous payment of retirement benefits, justifies the modeling choice of agents with intermediate consumption. The dynamic nature of the problem then arises by considering the temporal dimension of liquidity. In this interpretation, liquidity represents the opportunity to trade, occurring randomly with intensity determined by the period over which the asset cannot be traded on average. I find that the effect of illiquidity on the portfolio allocations is economically significant when calibrated to major asset classes.

This paper then looks at the asset allocation problem from several different angles. First, I review the baseline dynamic model of portfolio choice with continuous trading (Merton, 1971) as a benchmark model whose implications are relevant for the investment in traditional asset classes, such as money market funds, bonds, and equity. Then I reformulate the problem by introducing illiquidity and investigate the properties of the resulting optimal solution. Finally, I apply the models to the SAA problem of long-term investors with access to illiquid private asset classes along with traditional liquid assets.

The framework allows a distinction to be made between strategic and tactical asset

[1] See for example Andonov et al. (2015, 2021); Giesecke and Rauh (2023); Andonov et al. (2023).

[2] The static mean-variance optimization offers little consideration of the issue, yet it continues to be dominant in the industry. See for example Kim et al. (2021) for a survey on the preferences of investors for portfolio construction methodologies.

allocation (TAA). The SAA allows the investor to factor in *ex-ante* the costs associated with illiquidity and to construct the optimal base weights accordingly. Thus, it captures the weights that an investor will return to at every opportunity when liquidity is available. TAA on the other hand captures the second-best allocation that can be reached when liquidity is available only for a subset of the assets in the portfolio. Inevitably, as I will show, the TAA will be dependent on the size of the allocation locked in illiquid holdings.

First of all, the model supports the intuition that investors should reduce the allocation to assets that cannot be traded frequently, all else equal. The reduction becomes economically significant for the expected waiting time to trade the asset of more than two years. This results in a notable reduction in the strategic allocation to illiquid assets. Consequently, the cost of illiquidity increases significantly after that point.

Second, between trading opportunities, as the allocation moves away from the strategic base, investors optimally reduce the fraction of *liquid* risky assets in favor of cash. This dampens the overall volatility in liquid wealth and as a result the volatility in the consumption rate. Numerically, I find that this spillover effect of illiquidity on liquid holdings is small. For reasonable calibration the agent can rebalance the portfolio before entering a liquidity drain and before consumption needs to be cut significantly. Yet, illiquidity introduces a fat left tail in the distribution of liquid asset allocations, indicating that tactically in some occasions the investor may need to reduce significantly the allocation to liquid risky sub-portfolio.

Further analysis reveals a significant reduction in welfare for expected trading friction larger than five years. This can be attributed both to the decreased diversification potential of illiquid assets, and the higher risk of liquidity exhaustion before a trading opportunity arises. As a result, diversification benefits from low correlation between liquid and illiquid assets are shown to be less effective when illiquidity friction exists.

The paper continues as follows: Section 2 identifies the main gaps in the academic literature that this paper addresses; Section 3 outlines model, first providing the continuous trading model that will serve as a benchmark, and then builds on it to address illiquid assets; Section 4 shows the key properties of the model when applied on two synthetic identical assets, one of which is illiquid; Section 5 finally applies the model to analyst expectations data on private and public asset classes that long-term investors can typically access.

2 Related Literature

First of all, this paper relates to the wider literature on dynamic portfolio choice with trading frictions, which can broadly be classified along three main dimensions: transaction costs (Zabel, 1973; Magill and Constantinides, 1976; Gennotte and Jung, 1994; Boyle and Lin, 1997), delayed execution (Longstaff, 2001), and inability to access a market (Miklós and Ádám, 2002; Ang et al., 2014; Jansen and Werker, 2022), with each dimension having specific implications on the optimal strategies recommended for investors.

In the first dimension, asset liquidity can be accessed by paying a transaction cost. This approach to illiquidity has been widely explored in the earlier literature. The studies in this area generally find that with the transaction costs it becomes optimal for agents to trade only if the illiquid asset allocation moves outside of a no-trade region. Based on this finding, Constantinides (1986) argues that in general equilibrium the overall trading in the asset subject to trading costs will be significantly reduced even when the cost is modest. By trading smaller amounts and less frequently agents will thus be able to accommodate the transaction cost without sacrificing consumption. This creates market illiquidity while agents still do not require significant liquidity premia to compensate for the friction. This argument is debated for example by Dai et al. (2011) who find that when position constraints exist in combination with trading costs, this may give rise to significantly larger liquidity premia.

Later studies emphasize that holding liquid risky assets along with illiquid ones creates hedging and diversification motives. This reduces the utility cost of the transaction friction and lowers the impact of illiquidity on the asset allocation (Buss et al., 2015; Bichuch and Guasoni, 2018).

Longstaff (2001) explores the second dimension, where illiquidity appears as friction on the quantity that can be traded per period. In his setup, a constraint exists on the speed with which trades can be executed, so larger trades are executed slowly at a known rate. By waiting for a deterministic period, the agent is then subject to price risk without complete control over the allocations in the portfolio and has to hedge against both expected and unexpected portfolio weight changes. In contrast to the previous set of studies, they find that the agents do not face a no-trade region in their allocation.

This paper can be situated across the third dimension of illiquidity. In an early attempt to handle liquidity, Miklós and Ádám (2002) approximate the inability to trade an asset immediately by introducing a deterministic lag between the time an order is placed on the market and the time a trade takes place. Ang et al. (2014) further extend this idea by adding uncertainty in the waiting time. A similar approach, with uncertain trading time under a limited horizon setup, is applied by Jansen and Werker (2022) to estimate the potential cost of illiquidity for several illiquid asset classes.

The idea of random trading opportunities is closely related to the way OTC markets function. The search frictions associated with OTC markets typically imply that the time needed to find a trading counterparty, and thus the time between market transactions, will be stochastic (Diamond, 1982; Duffie et al., 2004). This also holds for the private assets that I consider here and which are exclusively OTC traded.

In general, the literature has identified several market imperfections that can exacerbate liquidity (Vayanos and Wang, 2012). I do not focus on the reasons for which illiquidity may persist, but still, it may be worthwhile to point out that each of the following market imperfections is present in the market for private assets, thus justifying the modeling approaches that I consider. First, clientele effects exist, which allow only investors with sufficient capital or expertise to trade, especially when participation costs and transaction costs are present and agents face charges for trading and monitoring market movements. Second, imperfect competition may allow large players to exert market power and may push out smaller investors causing a strain on liquidity. Third, the presence of asymmetric information may motivate buyers to leave the market. In addition, funding constraints, such as the low access to funding for investors may exacerbate illiquidity further (Brunnermeier and Pedersen, 2009). Also, search frictions, especially in OTC markets, typically create a decentralized network structure of the market where investors need to spend time looking for an appropriate trading counterparty to the trade as in Duffie et al. (2005).

I also relate to the literature on liquidity premia by evaluating the return surcharge that a utility-maximizing investor is willing to accept (in utility-equivalent terms) in order to convert the illiquid asset to a liquid one with otherwise comparable risk-return properties. Even though I do not consider the risk premia in a general equilibrium framework, the model still can quantify the extra return that investors would require to accept holding an illiquid asset. This is typically referred to as the shadow cost of illiquidity (Longstaff, 2001; Jansen and Werker, 2022).

Empirically, there is an academic debate on how large liquidity premia are and if they even exist given biases endemic to illiquid asset classes (Ang, 2011) and the difficulty of establishing a reliable observational measure of illiquidity even for publicly traded assets (Goyenkoa and Trzcinkab, 2009). Our approach provides a theoretically justified link between the severity of the liquidity friction and the size of the corresponding premium.

Overall, the liquidity premia literature tends to focus on market microstructure.[3] Amihud and Mendelson (2015) defines two channels, common in these studies, through which illiquidity may affect asset returns: the level effect (when an asset-specific characteristic leads to an additional market premium), and the risk effect (when an asset's returns are sensitive to market-wide liquidity shocks).

There are, however, very few studies that examine the effect of illiquidity on risk premia of private asset classes. Korteweg and Westerfield (2022) provide an overview of the main challenges in terms of data transparency, performance measurement, and investing in private asset classes. In terms of liquidity, Franzoni et al. (2012) find evidence that private equity shares the same liquidity factor as public equity, reducing the funds' alpha and the diversification potential typically expected by investors. Jansen and Werker (2022) apply a portfolio choice model with liquidity costs for investors with a limited horizon, calibrating the model to regulatory data of alternative investment holdings. I overcome the difficulty of estimating the actual return and risk characteristics on illiquid asset classes by directly employing forward-looking expected return and risk data from JP Morgan (2022). Giommetti and Sorensen (2021) apply a portfolio choice model to the specific features of a private equity's cash inflows and outflows. The goal here however is different. I model the allocation to the asset class as a whole, from the point of view of an institutional investor's SAA, rather than from the perspective of a particular fund's limited partner.

I also extend the literature on SAA for institutional investors[4] by explicitly incorporating a liquidity friction as a latent Poisson factor in the spirit of Ang et al. (2014). The approach can be related to studies of dynamic portfolio choice with jump risk (Wu, 2003; Liu et al., 2003). The main difference with the current approach is that at the moment the Poisson jump occurs, I allow for the decision-maker to access liquidity on the private asset and to reset the portfolio allocations to optimality.

Finally, I relate to the literature which uses numerical approaches to solve the dynamic portfolio choice model (Rust, 1996; Cai et al., 2013; Cong and Oosterlee, 2017).

3 Model

3.1 Investor Preferences and Evolution of Market Prices

The information set in the model follows a standard structure with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the vector of independent standard Brownian motions $Z_t =$

[3]For example within equity (Amihud and Mendelson, 1986; Pastor and Stambaugh, 2003; Acharya and Pedersen, 2005), corporate bonds (Bongaerts et al., 2012), government bonds (Amihud et al., 2005), CDS derivatives Qiu and Yu (2012) and crypto-currency (Brauneis et al., 2021) markets among other classes.

[4]See for example Campbell et al. (2004); Cochrane (2022).

(Z_{1t}, Z_{2t}) on $\mathbb{R}^n$. The agent has a CRRA utility over the consumption amount C_t through time:

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$

with $\gamma > 0, \gamma \neq 1$ as the coefficient of relative risk aversion, where higher γ implies a higher risk aversion.

Denoting the instantaneous risk-free rate as r, the price B_t of a risk-free asset follows the process:

$$dB_t/B_t = rdt \tag{2}$$

In addition, there are n risky assets on the market with prices collected in the vector $\boldsymbol{S}_t$, and their return $\boldsymbol{dS}/\boldsymbol{S}$ evolves as a multivariate Geometric Brownian Motion process such that:

$$\begin{aligned}\frac{\boldsymbol{dS}_t}{\boldsymbol{S}_t} &= \boldsymbol{\mu}dt + \boldsymbol{\sigma}\boldsymbol{dZ}_t \\ &= (r\mathbb{1} + \boldsymbol{\sigma}\boldsymbol{\lambda})dt + \boldsymbol{\sigma}\boldsymbol{dZ}_t\end{aligned} \tag{3}$$

where $\mathbb{1}$ stands for a n-dimensional column vector of ones, $\boldsymbol{dZ}_t$ is a vector of n independent Brownian motions supported by probability space $(\Omega, \Im, \mathbb{P})$; $\boldsymbol{\mu}$ is a $n \times 1$ vector of expected returns; $\boldsymbol{\sigma}$ is $n \times n$ matrix holding the sensitivity of the risky asset returns to the Brownian uncertainties; $\boldsymbol{\lambda} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbb{1})$ is the price of risk.

3.2 Benchmark Liquid Market Case

First, consider the benchmark case where all assets are continuously marketable.[5] Then, a representative investor holds a liquid financial portfolio out of which she can withdraw (consume) continuously the rate c_t. The agent has control over the asset allocation and decides over the risky asset investment proportions π_t. The wealth dynamics are then determined by the return of the portfolio net of the consumption rate:

$$\frac{dW_t}{W_t} = (r + \boldsymbol{\pi}_t'(\boldsymbol{\mu} - r\mathbb{1}) - c_t)dt + \boldsymbol{\pi}_t'\boldsymbol{\sigma}\boldsymbol{dZ}_t \tag{4}$$

Over time, subject to the wealth dynamics of (4), the agent optimizes her expected lifetime utility:

$$V(W_t) = \sup_{(\pi_s, C_s)} E_t \int_t^\infty \mathrm{e}^{-\beta(s-t)} u(C_s) ds \tag{5}$$

giving rise to the indirect utility of wealth function $V(W_t)$.

Applying the Bellman principle, we can derive the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}&\mathcal{L}^C + \mathcal{L}^\pi - \beta V = 0 \\ &\quad \mathcal{L}^C = \sup_{C_t} \Big\{ u(C_t) - C_t V_W \Big\} \\ &\quad \mathcal{L}^\pi = \sup_{\boldsymbol{\pi}_t} \Big\{ \left(r + \boldsymbol{\pi}_t'(\mu - r\mathbb{1})\right) V_W W_t + \frac{1}{2} V_{WW} W_t^2 \boldsymbol{\pi}_t' \boldsymbol{\Sigma} \boldsymbol{\pi}_t \Big\}\end{aligned} \tag{6}$$

[5] This set up has been widely explored in the literature, so here I only provide a high level discussion. Annex (A.1) provides the details behind the derivations in this subsection. For a thorough overview of portfolio choice, asset pricing and dynamic programming in continuous time you can refer to Duffie (2001); Back (2010); Munk (2014).

where V_W and V_{WW} are the first- and second-order partial derivatives of the value function with respect to wealth.

Equation (6) allows us to determine the optional allocation by optimizing separately $\mathcal{L}^{\pi}$ and the optimal consumption over time from optimizing $\mathcal{L}^{C}$.

First, the optimizing over π_t implies:

$$\boldsymbol{\pi} = -\frac{V_W}{W_t V_{WW}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\lambda} \tag{7}$$

where $\boldsymbol{\pi}$ is the vector of optimal risky allocations. The agent then will thus invest a fixed fraction $-\frac{V_W}{WV_{WW}} \mathbb{1}'(\boldsymbol{\sigma}')^{-1}\boldsymbol{\lambda}$ in risky assets and the rest will be allocated to the risk-free asset. The fraction $-\frac{V_W}{WV_{WW}}$ defines her relative risk tolerance given their indirect utility $V(W)$.The optimal risky allocation then is determined through the interaction of the investor's risk tolerance with the price of risk of each specific asset in the investment universe.

Second, we can derive optimal consumption[6]as:

$$u'(C_t) = V_W \implies C_t = (V_W)^{-\frac{1}{\gamma}} \tag{8}$$

The value function can be determined in closed form as

$$V(W_t) = \left(\frac{1}{c}\right)^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma} \tag{9}$$

As a result, one can then see that the optimal consumption rate (C_t/W_t) is fixed over time and given by

$$c = \frac{\beta + r(\gamma - 1)}{\gamma} + \frac{1}{2}\frac{\gamma - 1}{\gamma^2}\|\boldsymbol{\lambda}\|^2 \tag{10}$$

3.3 Introducing an Illiquid Asset

Assume now that a liquid market does not exists for the n-th asset in specification (3), meaning that uncertainty exist if the investor can trade this asset over time. The asset's fundamental value still continues to evolve through specification (3), but the investor may over periods of stochastic length be unable to tap into this value.

Following Ang et al. (2014), we formalize the liquidity uncertainty by assuming that a trading opportunity for the asset arrives with the intensity η of a Poisson process N_t, which is independent of the Brownian motions. Poisson arrivals are a common way to model search frictions, cf. Diamond (1982). A jump in N_t indicates that a liquidity opportunity for the asset has materialized and the investor can freely rebalance the portfolio to her strategic objectives.[7]

I assume that the illiquid asset cannot be used as collateral to issue riskless debt. The assumption ensures that the illiquidity cannot be circumvented by pledging the illiquid

[6]This is the Envelope Theorem, implying that at optimality the marginal utility from consuming a little more needs to be the same as the marginal value of investing a little more.

[7]Note one major difference between the Brownian Motion used to drive the asset return dynamics and the Poisson Process used to drive the liquidity dynamics: in the former, the size of the movement and thus its variance is dependent on the horizon, while in the latter, the probability of occurrence is horizon-dependent.

asset and issuing debt. This is also empirically valid, as the private assets are not likely to be accepted as collateral.

The investor's total wealth Q_t now consists of liquid asset holdings W_t (into risky and riskless assets), and illiquid risky holdings X_t. The only income the investor receives is from the capital gains on the invested wealth. She then controls several aspects of her asset allocation: the liquid portfolio composition (allocation weights $\boldsymbol{\theta}_t$ into risky liquid asset); and the transfer amount dI_t from illiquid to liquid wealth, which is possible only when a trading opportunity arises through the shock N_t. Furthermore, the investor decides how much to consume through the consumption (withdrawal) rate c_t. Consumption, however, is only possible out of liquid wealth. Then we can write the wealth dynamics of the liquid and illiquid wealth as follows:

$$\begin{aligned} dW_t/W_t &= (r + \boldsymbol{\theta}_t'(\mu_{1:n-1} - r\mathbb{1}_{1:n-1}) - c_t)dt + \boldsymbol{\theta}_t'\boldsymbol{\sigma}_{n-1}\boldsymbol{dZ}_t - dI_t/W_t \\ dX_t/X_t &= \mu_n dt + \boldsymbol{\sigma}_n \boldsymbol{dZ}_t + dI_t/X_t \\ dQ_t &= dW_t + dX_t \end{aligned} \tag{11}$$

where the subscripts $1:n-1$ and n indicate the first $n-1$ rows, or respectively the n-th row of the corresponding matrix or vector, specified in (3).

The value function for this problem can be written as

$$V(W_t, X_t) = \sup_{\theta_s, dI_s, c_s} E_t \int_t^\infty \mathrm{e}^{-\beta(s-t)} u(C_s) ds \tag{12}$$

Denote the share of illiquid wealth as

$$\xi_t = \frac{X_t}{Q_t}$$

Using the homogeneity properties of the CRRA utility, the reduced-form value function can be written as

$$H(\xi) \equiv V((1-\xi), \xi) = \left(\frac{1}{Q_t}\right)^{1-\gamma} V(W_t, X_t) \tag{13}$$

where $H(\xi_t)$ is a finite, continuous and concave function, maximized at ξ^* for $\xi \in [0, 1)$ (cf. Ang et al. (2014)).

Whenever liquidity is available, the investor is free to move to the top of the $H(\xi_t)$ curve by returning to the optimal proportion of liquid wealth:

$$\xi^* = arg \max_{\xi} H(\xi)$$

which is equivalent to choosing a transfer amount from and to illiquid wealth, such that

$$dI_\tau = (\xi^* - \xi_{\tau^-})W_{\tau^-}$$

where ξ_{τ^-} is the initial illiquid wealth fraction just before rebalancing occurs. From that point of view, ξ^* can be interpreted as the target strategic allocation to the illiquid asset. Whenever the asset stays illiquid, the agent is stuck with sub-optimal ξ_t.

As a result, we can transform the optimization problem of (12) from one of finding the value function $V(W_t, X_t)$ itself to solving for the function $H(\xi_t)$ and its maximum point ξ^*.

3.3.1 The HJB Equation with Three Assets

To gain intuition into the solution, we simplify the problem and look at a three asset case with a liquid risk-free, and liquid risky and illiquid risky asset. Then we have the set-up:

$$\boldsymbol{\mu} = \left[\mu_1, \mu_2\right]^\intercal, \boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}, \boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

To ensure that the illiquid asset class is viable for investment and given the illiqudity property, I assume that its Sharpe ratio is at least as large a as that of the liquid asset:

$$\frac{\mu_2 - r}{\sigma_2} \geq \frac{\mu_1 - r}{\sigma_1}$$

The wealth dynamics then will evolve as

$$\begin{aligned} dW_t/W_t &= (r + (\mu_1 - r)\theta_t - c_t)dt + \theta_t\sigma_1 dZ_{1t} - dI_t/W_t \\ dX_t/X_t &= \mu_2 dt + \sigma_2\rho dZ_{1t} + \sigma_2\sqrt{1-\rho^2}dZ_{2t} + dI_t/X_t \end{aligned}$$

In Annex A.3 I then derive the HJB equation governing optimal consumption and allocation:

$$\begin{aligned} &\mathcal{L}^C + \mathcal{L}^\theta + \mathcal{L} = \beta V \\ &\qquad \mathcal{L}^C = \sup_{C_t} \left\{ u(C_t) - C_t V_W \right\} \\ &\qquad \mathcal{L}^\theta = \sup_{\theta_t} \left\{ (r + \theta_t(\mu_1 - r\mathbb{1}))\, V_W W_t + \frac{1}{2} V_{WW} W_t^2 \theta_t^2 \sigma_1^2 + V_{WX} W_t X_t \theta \sigma_2 \sigma_1 \rho \right\} \\ &\qquad \mathcal{L} = V_X \mu_2 + \frac{1}{2} V_{XX} X^2 + \eta(V^* - V) \end{aligned} \tag{14}$$

with V^* the point at which the value function jumps once the Poisson shock hits and the agent changes the illiquid-to-total wealth ratio back to the strategic ξ^*.

As a result total wealth can be canceled out and the optimal consumption rate (out of total wealth) c_t can be expressed as[8]

$$c(\xi_t) = \Big((1-\gamma)H(\xi_t) - H'(\xi_t)\xi_t \Big)^{-\frac{1}{\gamma}} (1 - \xi_t)^{-1} \tag{15}$$

In contrast to the fixed Merton optimal consumption rate in (10), the consumption rate between trading events now is state dependent and varies with ξ_t.

The optimal investment in the risky liquid asset can be derived correspondingly:

$$\theta(\xi_t) = \frac{\mu_1 - r}{\sigma_1^2} \underbrace{\left(-\frac{V_W}{V_{WW} W_t} \right)}_{\Phi(\xi_t)} + \frac{\sigma_2\rho}{\sigma_1} \underbrace{\left(-\frac{V_{WX} X_t}{V_{WW} W_t} \right)}_{\Psi(\xi_t)} \tag{16}$$

[8]To see that, note that at optimum, the marginal utility of liquid wealth is equal to the value of a marginal change in liquid wealth, such that

$$\begin{aligned} u'(C_t) &= V_W(W_t, X_t) \\ \iff C_t^* &= I_{u'}\left(V_W(W_t, X_t)\right) \equiv \left(V_W(W_t, X_t)\ \right)^{-\frac{1}{\gamma}} \end{aligned}$$

In the CRRA case in particular, $V_w(W_t, X_t) = W_t^{-\gamma}\left((1-\gamma)H(\xi_t) - \xi_t H'(\xi_t)\right)$ and $u'(c_t W_t) = u'((1 - \xi_t)W_t) = (1-\gamma)(cW)^{-\gamma}$.

The reaction function $\theta(\xi)$ when trading opportunity in the illiquid asset is not available, will form the tactical allocation as a reaction to the illiquid wealth share in the portfolio.

Note that the optimal investment in the liquid risky asset can be split into two parts. First, the *investment demand* in the liquid risky asset is given by

$$\frac{\mu_1 - r}{\sigma_1^2}\Phi(\xi_t)$$

where it can be seen that the function $\Phi(\xi_t) = -\frac{V_W}{V_{WW}Q_t}\frac{1}{1-\xi_t}$ is decreasing in ξ_t, keeping in mind that $-\frac{V_W}{V_{WW}} \geq 0$. As a result, even with zero correlation between the two risky assets, the optimal investment allocation in the risky liquid asset will still be state-dependent, and decreasing in the share of current illiquid asset holdings.

Second, the *hedging demand* term is represented by

$$\frac{\sigma_2 \rho}{\sigma_1}\Psi(\xi_t)$$

where $\Psi(\xi_t) = -\frac{V_{WX}}{V_{WW}}\left(\frac{\xi_t}{1-\xi_t}\right)$ again can be shown to be decreasing in $\xi_t \in (0,1)$, as due to the convexity of the value function $-\frac{V_{WX}}{V_{WW}} \leq 0$. If the correlation between the two assets then is negative, increases in the share of the illiquid asset will increase the hedging demand for the liquid asset, counterbalancing the investment demand effect. For positive correlation, however, the hedging and the investment demand will be going in the same direction, both decreasing the share of liquid risky investments when the share of illiquid wealth increases.

3.3.2 Discretization and Numerical Approaches

Solving the problem defined so far requires numerical approaches in order to determine the reduced-form value function. This section discusses briefly the discretization and solution steps. In Annex (B.1) I show the full details.

In the discretized version of the problem, we can focus on the probability p the agent will be able to trade the illiquid asset next period, which is in fact a function of the Poisson process arrival time η. We know that a Poisson process has stationary and independent increments, where the number of events that occur during any time increment of length Δt is Poisson distributed with mean $\eta\Delta t$.[9] We can then relate the illiquid asset's trading probability as the Poisson probability of having at least one trading event over time period Δt, given that the average time to wait for a trading opportunity over a year is $1/\eta$, and the average number of trading opportunities per year correspondingly is η. In that case:[10]

$$p = 1 - \mathrm{e}^{-\eta\Delta t} \tag{17}$$

[9]Formally, for any $t \geq 0, \Delta t > 0$, the probability of n events occurring can be written as:

$$P(N_{t+\Delta t} - N_t = n) = \mathrm{e}^{-\eta\Delta t}\frac{(\eta\Delta t)^n}{n!}, \quad n = 0, 1, ...$$

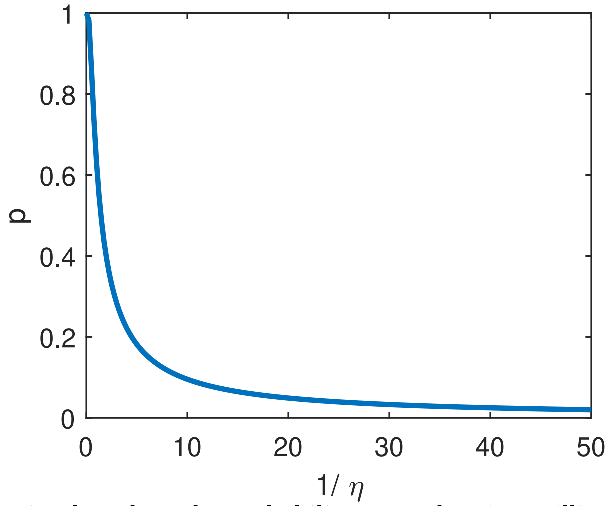
[10]Note that

$$p = P(N_{t+\Delta t} - N_t \geq 1) = 1 - P(N_{t+\Delta t} - N_t = 0)$$

which leads to the provided formula

Figure 1 illustrates this functional relationship over an annual horizon.

Figure 1: Relationship between p and η



The plot shows the connection based on the probability to trade p in an illiquid asset within a year and η, the average waiting time in years between trades.

Then, refering back to the illiquid investment problem, with with probability p the agent will be able to trade the illiquid asset next period and will bring the portion of illiquid wealth to the desired optimal level ξ. This is done by setting the transfer dI_t between liquid and illiquid wealth to accommodate optimal asset allocations, so:

$$dI_t = \begin{cases} \xi^* Q_t - X_{t-} & \text{with probab. } p \\ 0 & \text{with probab. } 1-p \end{cases}$$

where X_{t-} is the level of illiquid holdings just before rebalancing takes place.

The investor will set $\xi_{t+\Delta t} = \xi^*$ whenever trading is possible. If trading is not possible, the agent cannot rebalance and is stuck with sub-optimal levels of illiquid holdings $\xi_{t+\Delta t} = X_{t+\Delta t}/Q_{t+\Delta t}$, and the ratio will float away from last period's value as the prices of the two risky assets move erratically. This is illustrated in Figure 2, which makes a distinction between the state with liquidity, occurring with probability p and the state without market liquidity for the asset.

Following Bellman's principle of optimality, we can write the discretized version of the value function (12) as

$$V(W_t, X_t) = \max_{(\theta_t, dI_t, c_t \in \mathcal{A})} \{u(C_t)\Delta t + \delta E_{W_t,X_t}[V(W_{t+\Delta t}, X_{t+\Delta t})]\} \tag{18}$$

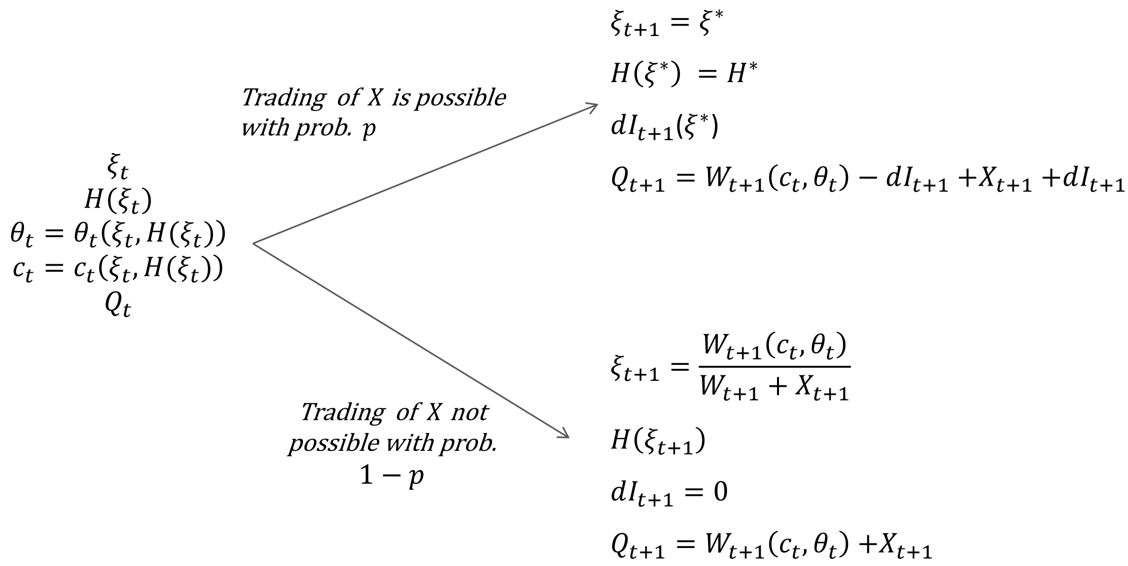
with $\delta = \mathrm{e}^{-\beta \Delta t}$ as the discrete-time discount factor.

As indicated earlier, knowing the reduced form value function $H(\xi)$ allows us to find the optimal ratio of illiquid to total wealth ξ^* as the maximizing value for that function. Our goal is then to write the Bellman equation in a way which allows us to find ξ^* and $H(\xi_t)$.

Factoring out total wealth, we can rewrite the Bellman equation in terms of the reduced form value function as

$$H(\xi_t) = \max_{(\theta_t, \xi_t, c_t)} \left\{u(c_t(1-\xi_t))\Delta t + \delta \left(pH^* E_{\xi_t=\xi^*}\left[R^{1-\gamma}_{q,t+\Delta t}\right] + (1-p)E_{\xi_t \neq \xi^*}\left[R^{1-\gamma}_{q,t+\Delta t} H(\xi_{t+\Delta t})\right]\right)\right\} \tag{19}$$

Figure 2: State Transition Dynamics



This chart illustrates the dynamics behind the optimal choice problem. In the coming period, the illiquid asset share floats freely to $\xi_{t+\Delta t}$. With probability p the agent can trade in the illiquid asset and can set it back to the optimal target of ξ^* by maximizing the known function $H(\xi)$. With probability $1-p$ the agent cannot trade and is stuck with the illiquid asset share of $\xi_{t+\Delta t}$.

subject to the laws of motion

$$\begin{aligned} R_{q,t+\Delta t} &= (1-\xi_t)R_{w,t+\Delta t} + \xi R_{x,t+\Delta t} \\ \xi_{t+\Delta t} &= \xi_t \frac{R_{x,t+\Delta t}}{R_{q,t+\Delta t}} \end{aligned} \tag{20}$$

where $R_{q,t+\Delta t}$ is the discretized growth of total wealth in the next period, $R_{w,t+\Delta t}$ is the growth of liquid wealth net of current consumption, $R_{x,t+\Delta t}$ is the growth of illiquid wealth, and $\xi_{t+\Delta t}$ is the holdings ratio given that rebalancing is not possible (cf. B.1).

The discretized Bellman equation can then be solved through value function iteration combined with standard numerical techniques. In Appendix B.2 we show the algorithm used for the purpose.

3.3.3 The Certainty Equivalent and Utility Loss

Apart from determining the optimal investment and consumption strategies, we want to quantify the cost of illiquidity for each asset class in utility-equivalent terms. For this purpose, first define the lifetime utility value generated by guaranteed continuous consumption stream $\overline{C}$ as:

$$\int_t^\infty \mathrm{e}^{-\beta s} u(\overline{C}) ds = \frac{1}{\beta} u(\overline{C})$$

The certainty equivalent consumption (CEC) then is defined as the as the consumption which makes agents indifferent between risk-free stream $\overline{C}$ and uncertain consumption underlying a certain consumption strategy. It is determined by equating the cumulative utility associated with risk-free consumption to the indirect utility associated with a particular risky consumption stream $\{C_s\}_{s\in[t,\ldots,\infty)}$:

$$CEC_t = I_u\left(\beta E_t \int_t^{\infty} \mathrm{e}^{-\beta s} u(C_s)ds\right) \tag{21}$$

where $I_u(.)$ stands for the inverse of the utility function. Noting that if C_s is determined through one of the optimizations outlined above, the term in the integral is in fact the value function: either (5) or (12). The we can used the value function determined in each optimization problem to value the CEC associated with that strategy. As a result, expressing the CEC as a fraction of total wealth, we find that, for the fully liquid Merton case with consumption rate c_M:

$$\frac{CEC_t^M}{W_t} = (\beta(1-\gamma)H_M)^{\frac{1}{1-\gamma}} = \beta^{\frac{1}{1-\gamma}}\left(\frac{1}{c_M}\right)^{\frac{1}{1-\gamma}}$$

where $H_M = V(W_t)/W_t$ is the reduced form value function for the Merton case.

For the illiquid case, the CEC will dependent on the current share of illiquid wealth:

$$\frac{CEC_t^{il}(\xi)}{Q_t} = (\beta(1-\gamma)H(\xi_t))^{\frac{1}{1-\gamma}} \tag{22}$$

We can then define the cost of liquidity $L(\xi)$ as the CEC loss associated with a lack of liquidity. It is measured as the percentage of CE consumption the agent would be willing to give up in order to make the second risky asset completely liquid. In particular I have

$$L(\xi_t) \equiv 1 - \frac{CEC_t^{il}(\xi)}{CEC_t^{M2}} = 1 - \left(\frac{H(\xi_t)}{H_{M2}}\right)^{\frac{1}{1-\gamma}} \tag{23}$$

where CEC_t^{M2} is determined for the Merton two-asset case. Note that the utility loss is again state-dependent through ξ_t.

It can be shown that $L(\xi_t)$ is also the percentage loss on current total wealth that individuals are willing to take in order to make their wealth fully liquid. For the individual to be indifferent between holding fully liquid or holding illiquid wealth, the value function in the Merton case with two liquid assets (V^{M2}), adjusting for the acceptable discount, has to be equal to the value function in the illiquid case:

$$\begin{aligned} V^{M2}\Big(Q_t(1-L(\xi_t))\Big) &= V(W_t, X_t) \\ \implies L(\xi_t) &= 1 - \left(\frac{H(\xi_t)}{H_{M2}}\right)^{1-\gamma} \end{aligned}$$

The last term is precisely the expression in (23).

In the following two sections, I relate the two dynamic portfolio choice models to the SAA process for long-term investors. First, Section 4 examines the model implications for a three-asset stylized example. Second, Section 5 uses the model to compare the optimal allocations with private asset classes to a traditional asset allocation consisting of equities and bonds, for which liquidity is not a concern from the perspective of long-term investors.

4 Stylized Example: Properties of the Illiquid Asset Allocation

Now, assume that the investment universe consists of two identical assets in terms of risk and return. The second asset, however, is subject to illiquidity risk. We can use the model to evaluate the optimal asset allocations and compare them to the fully liquid solution. This will shed also light on the illiquidity premia.

Overall, a risk-aversion coefficient $\gamma = 6$, as in the Merton two-asset case, under the given initial parametrization, this produces roughly 60% investment in risky assets and 40% in risk-free bonds which is a standard long-term investment strategy for moderately risk-averse agents. The rest of the input variables used in are given in Table 2a.

4.1 Optimal Allocations

First, we examine how the portfolio allocations are affected by the severity of the liquidity friction. Table 1 presents the effects of different average waiting times to trade $(1/\eta)$ on the SAA. When trading can occur up to a year on average, the optimal allocation with illiquidity is still close to the continuous trading case - around 30% is allocated to the second asset. As the trading friction intensifies ξ^* is significantly reduced. Figure 3a also illustrates the point – the optimal base allocation to the illiquid asset starts decreasing with a friction above one year and the cash cushion starts increasing. The higher cash buffers dampen any shocks to liquid wealth.

Table 1: Allocation (SAA), Consumption and Cost of Illiquidity

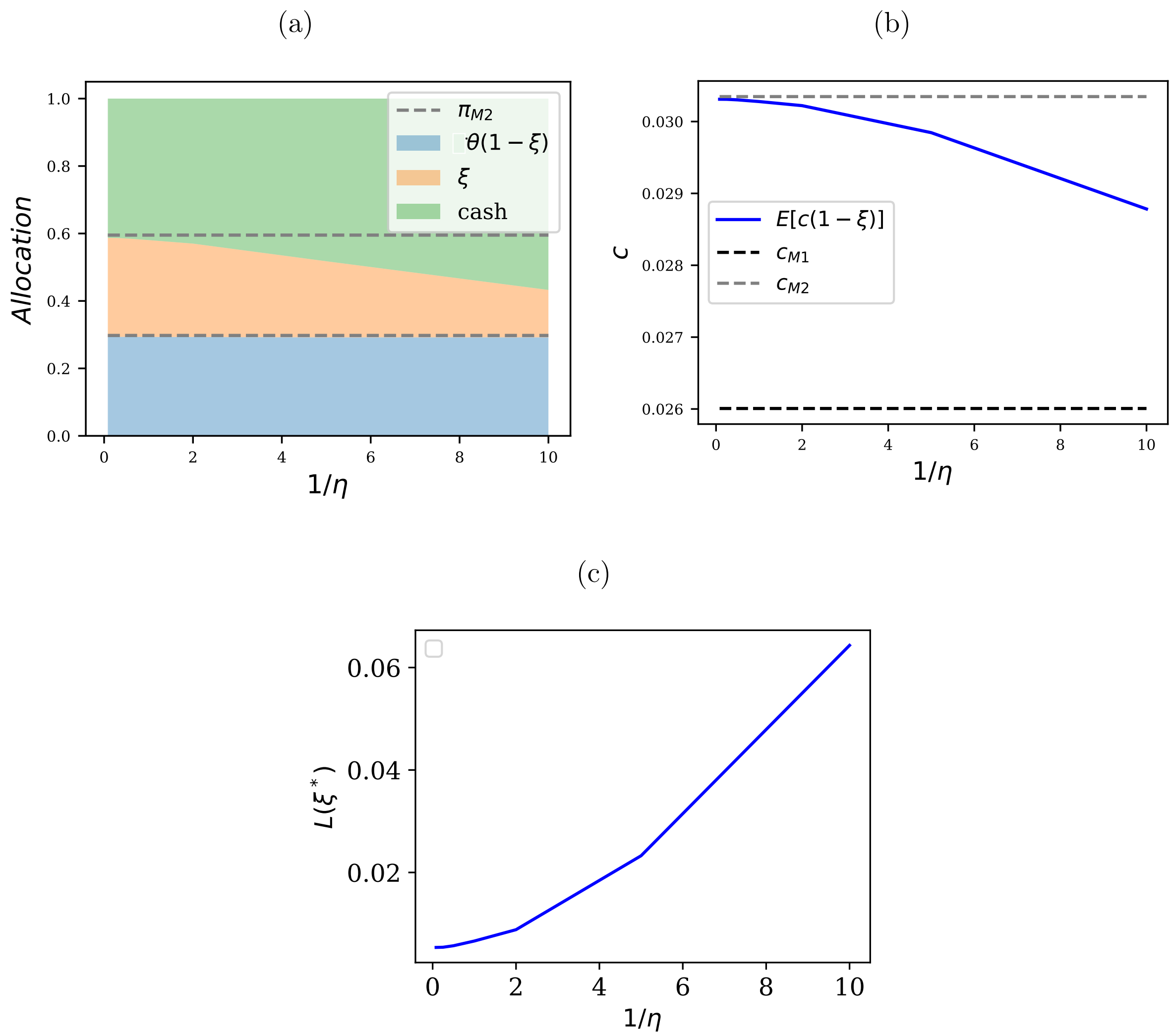
		Cost		Consumption		Allocation				Rabalancing	
η	p	$rp(\xi^*)$	$LC(\xi^*)$	$CEC(\xi^*)$	$E[c(\xi)(1-\xi)]$	$\theta(\xi^*)(1-\xi^*)$	$E[\theta(\xi)]$	ξ^*	$E[\xi]$	$E\left[dI^+/Q\right]$	$E\left[dI^-/Q\right]$
M1	-			2.53	2.60	29.76		29.76			
M2	-			3.04	3.03	29.76		29.76			
1/12	100.00	0.01	0.54	3.03	3.03	29.35	29.35	29.35	29.40	0.83	0.96
1/4	98.17	0.01	0.54	3.03	3.03	29.35	29.34	29.33	29.58	1.09	1.41
1/2	86.47	0.05	0.57	3.02	3.03	29.34	29.33	29.18	29.66	1.39	2.18
1	63.21	0.06	0.66	3.02	3.03	29.33	29.29	28.72	29.89	1.66	2.86
2	39.35	0.08	0.88	3.01	3.02	29.29	29.21	27.71	29.29	2.02	4.27
5	18.13	0.28	2.33	2.97	2.91	29.19	26.98	22.58	30.41	2.38	10.62
10	9.52	1.21	6.43	2.85	2.79	29.20	26.95	14.06	21.60	2.11	11.61

Note. This table shows the cost of liquidity and the sensitivity of optimal consumption and allocation to the strength of the liquidity uncertainty. The liquidity friction is quantified by the first two columns - trading interval and probability of being able to trade over the coming year. The first two lines stand for the continuous-trading Merton case with one and two assets (M1 and M2 respectively). All numbers are in percentage points.

As the investor is not able to trade continuously, the discrepancy between the base allocations ξ^*, and the average realized weights $E[\xi]$ is larger, the more illiquid the private asset. Occasionally, when the Poisson liquidity shock hits, the investor will be able to rebalance back to the base weights (ξ^* and $\theta(\xi^*)$) and avoid getting stuck with extreme levels of illiquid wealth. The rest of the time, she will be forced to hold sub-optimal allocations.

The average tactical (i.e. realized) allocation moves down less sharply with illiquidity than the optimal SAA, as seen in Table 1. It is not surprising then that the average withdrawal rates from illiquid wealth $E\left[dI^-/Q\right]$ become larger, the longer is the expected

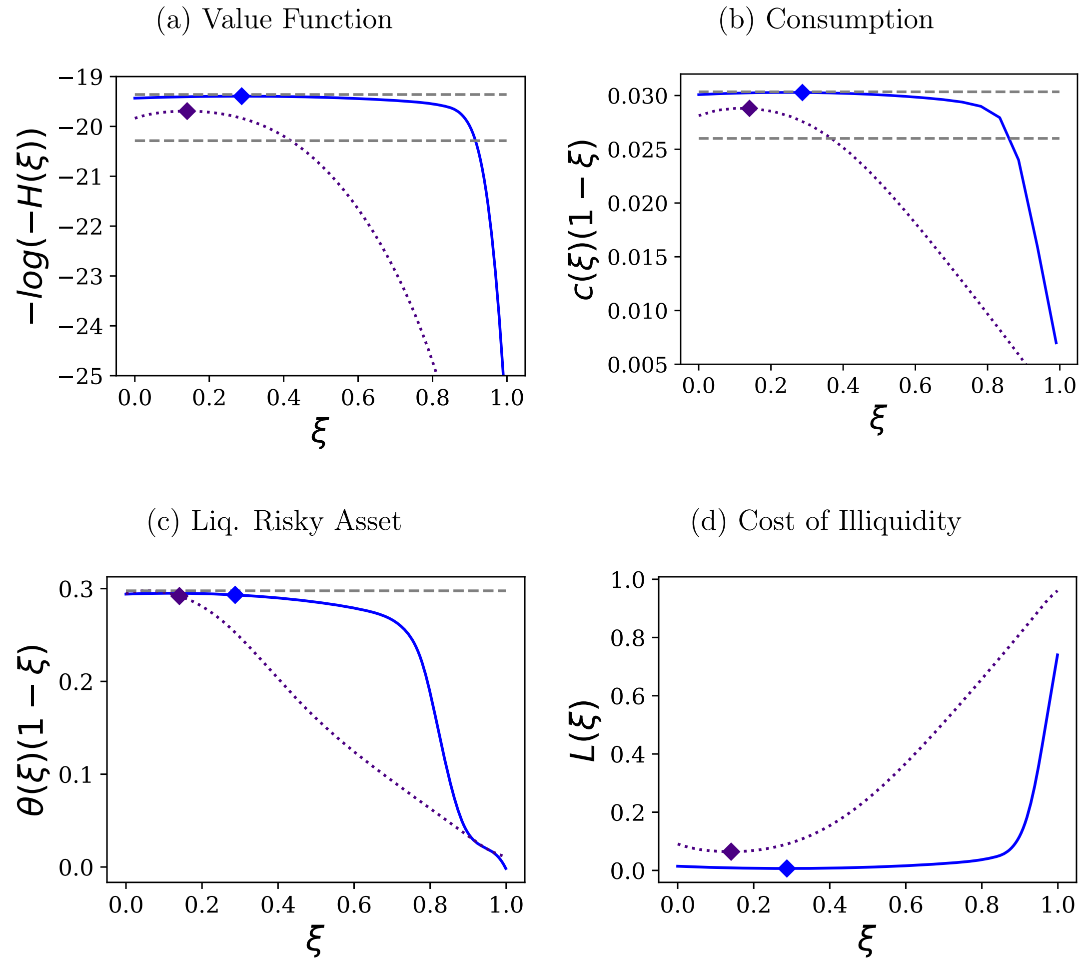
Figure 3: Allocation (SAA), Consumption and Welfare Loss



Note. This figure shows the effect of the trading friction (a) on the asset allocation as a function of the average waiting time to trade, (b) on the illiquid asset holdings as a function of the probability of being able to trade over the coming year, the dashed line indicating the Merton two-asset solution. Part (c) shows certainty equivalent consumption loss as percent to total wealth. The grey dashed lines stands for the two-asset continuous trading case, the black dashed line for the one-asset continuous trading case.

waiting time to trade, while the average investment rate $E\left[dI^-/Q\right]$ is not significantly affected.

Figure 4: Consumption and Allocation (TAA) Responses to Illiquid Asset Holdings



Note. This set of figures displays consumption and liquid asset holdings as a reaction to the illiquid asset allocation level. Each chart presents a one-year (solid curve), ten-year friction (dotted curve) and the continuous trading cases denoted by the flat dashed lines. Panel 4a shows the reduced form of the value function $H(\xi)$, compared to a Merton liquid market with one asset (bottom dashed line) and with two assets (top dashed line). Panel 4b shows optimal consumption. Panel 4c displays optimal liquid asset holdings. Panel 4d illustrates the certainty equivalent cost associated with a one-year and a ten-year friction. If a trading opportunity arises, the agent returns to the point indicated by a blue diamond in each of the charts.

During periods of illiquidity, the agents can only adjust consumption and investment in the liquid asset as a response to the stochastic share of illiquid wealth. The optimal allocation, subject to the inability to rebalance illiquid wealth in our setting represents the tactical allocation (TA), as the investor tries to dampen shocks to illiquid wealth by tactically adjusting between liquid assets. This intuition underlies Figures (4) where the (reduced) value function, consumption, and the liquid risky asset allocation are represented as response functions to the current share of illiquid wealth.

Figure 4c illustrates the TAA curves, the response of the liquid asset to movements in the illiquid asset share for a one-year and a ten-year friction. The Merton liquid case, the one- and two-risky-asset market solutions respectively, is presented by the two flat dashed lines. When the assets are fully liquid, the agent can rebalance continuously and allocations in one asset are insensitive to the current endowment in other assets. In an environment with a trading friction, the investor needs can rebalance back to SAA only when the opportunity arises and she can withdraw from illiquid wealth. The rest of the time only consumption and liquid risky holdings can be adjusted.

In the one-year illiquid case, the base illiquid wealth allocation ξ^* stands close to the two-asset case liquid solution of about 30% indicated by the diamonds on the graphs. The value function also stays close to that of the two-asset cases whenever illiquid wealth is below 80% of total wealth. Below that threshold, all one-year curves are flat, and the agent can finance more or less the same consumption rate as in the continuous trading case at any level of ξ.

In each chart, a gap between the Merton flat lines and the illiquid solutions forms sharply as illiquidity grows beyond that point and as the chance grows for the agent to end up in scenarios where liquid wealth is too low to finance consumption. In particular, agents need to prevent scenarios in which liquid wealth is exhausted before the next trading opportunity arises, so consumption and liquid risky investments are getting reduced preventively. Liquid asset investments are reduced sharply at the upper edge of the curve, which curtails the volatility of liquid wealth. In the case of a ten-year friction the same effect occurs much sooner on the ξ dimension. At around $\xi = 20\%$ the reaction curves of consumption and liquid investment start decreasing linearly.

4.2 Consumption and Welfare Losses

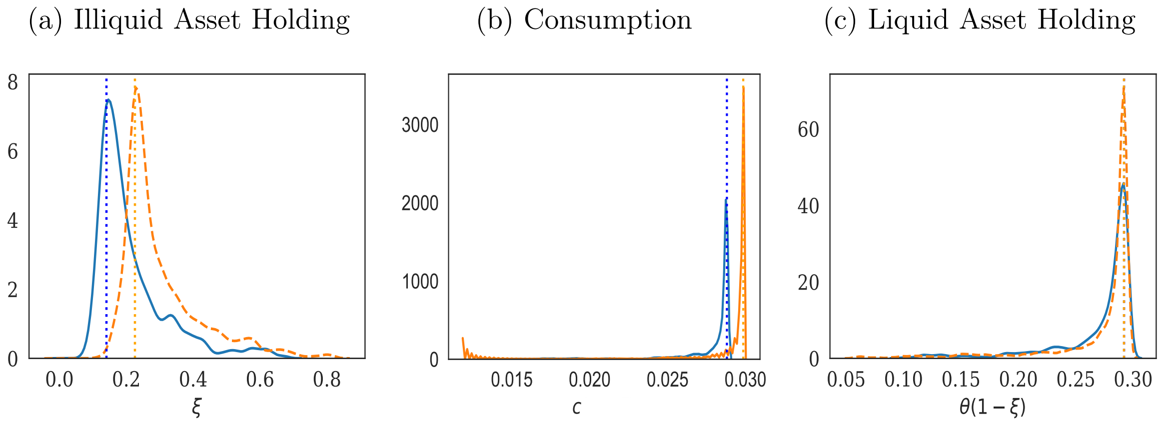
Table 1 shows the certainty equivalent consumption given that the allocation is in line with the SAA targets ($CEC(\xi^*)$). This serves as an indication of the risk-free equivalent of the consumption which can be financed by the risky portfolio. As we increase the expected waiting time between trades, clearly this reduces the equivalent consumption rate, indicating a loss of welfare measured on a risk-adjusted basis. Similarly, as the friction increases $L(\xi^*)$ indicates that the agents will be willing to give up larger portion of the initial wealth to make the illiquid asset fully liquid. In both cases, the welfare effects become economically significant for at least six month expected weighted weighting time to trade.

Figure 4d illustrates the full range of the welfare losses from a TAA point of view, i.e. when the investor cannot adjust back to SAA targets, as a function of of the current share of illiquid wealth. Then ξ^* defines the lower limit on these welfare losses. What we can observe is that for a one-year waiting time (the solid line), the TAA loss curve is very flat for most of the range over ξ_t, indicating that consumption does not react strongly to the share of illiquid assets, since the agent can rebalance often enough. By shifting the expected trading time, however, the welfare losses rise sharply once a certain threshold of ξ_t, after which the agents find it difficult to finance consumption. For the ten-year waiting time (dashed curve) the loss rises notably even for moderate deviations of ξ_t from ξ^*.

Overall, the welfare loss from the illiquid and the continuous-trading case can be attributed to two factors. First, higher liquidity friction implies a higher constraint on the ability of agents to finance their consumption, when liquid holdings decrease too much. Second, the lack of trading opportunity leads to lower diversification, as *ex-ante* the holdings in the illiquid asset are reduced. The longer the expected waiting time until trading can happen, the higher each of these two components to the welfare cost will be.

The illiquidity cost can also be translated into a liquidity premium – the spread over the illiquid asset's expected return in order to make it just as attractive as a liquid asset with otherwise the same risk characteristics. The column α_{lp} in Table 1 makes this point. For an asset that can be traded semi-annually on average, the liquidity premium is about $5bps$. The premium grows to above 1.21% for assets which can be expected to be traded ten years from now. This premium is again computed at the high point of the value curve and presents a lower bound on the possible observed losses.

Figure 5: Simulation Paths with Illiquidity

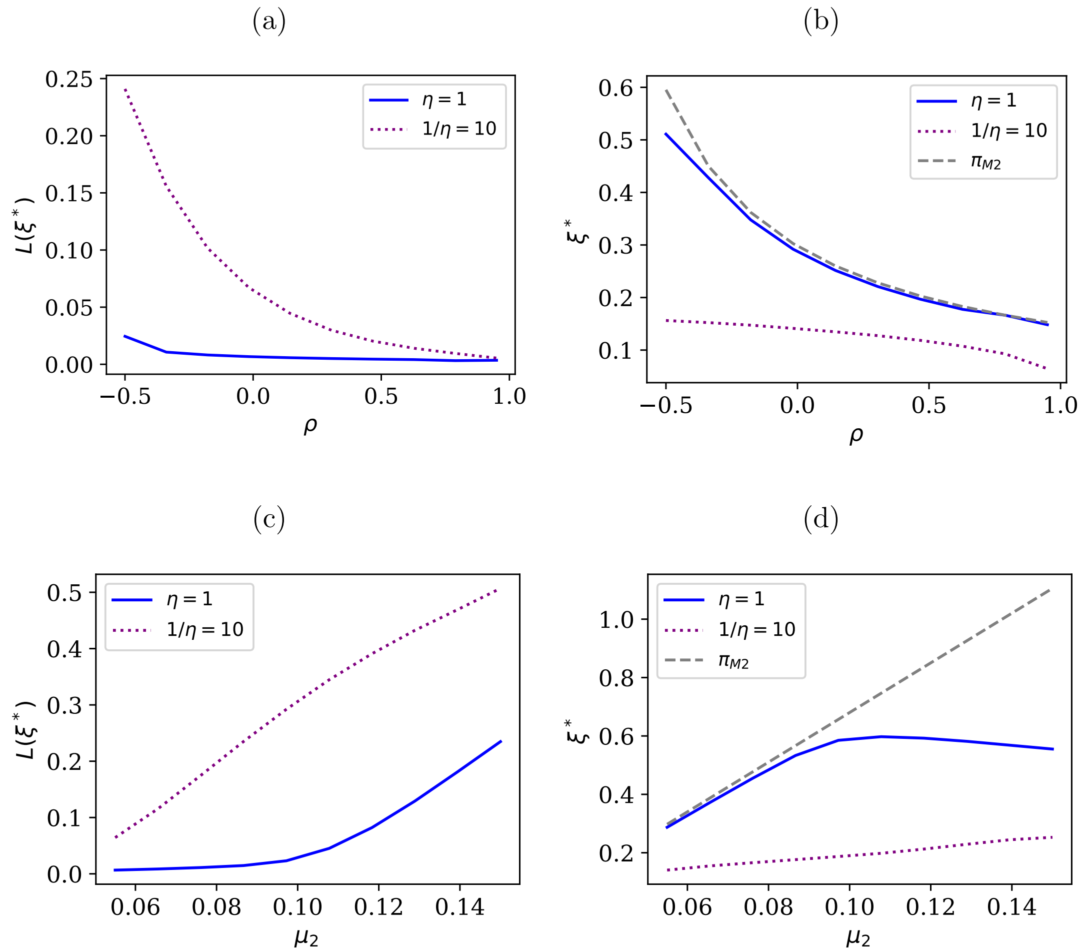


This set of charts show the corresponding density of the decision variables in the model. All fractions are expressed as a percentage of total wealth. Consumption and liquid wealth allocation vary with the level of illiquid asset holdings ξ. The dotted vertical line in each case shows the optimal values at the base allocation point when liquidity is available. The orange line here is for the one-year friction, the blue line is for the ten-year one.

The cost that illiquidity posts on long-term investors is also shown in Figure 5 where I simulate for the two assets a number of time paths of length 100 years and allow the investor to follow the optimal strategies discussed so far. It can be seen that consumption and wealth grow at a lower rate when the friction is higher. More severe illiquidity induces more conservative investment, which can also be traced back in the optimal composition of the portfolio in Figure 3a, stating that as the friction intensifies, the investor holds a higher proportion of the portfolio into cash as a cushion on the wealth volatility and illiquidity risk. A lower share of risky assets, in turn, hampers the investor's ability to build up wealth over time and consumption. I can see that in the case of the ten-year friction (purple), consumption grows at a slower pace compared to the case with a more liquid asset (blue).

Figure 3a also provides the simulated densities of the decision variables of the model. In (5a), the illiquid asset allocation exhibits a fat right tail and a tendency on average to float above the target level ξ^* indicated by the vertical dashed line. The expected return of the two risky assets is the same, still, the continuous consumption withdrawal from liquid wealth reduces its expected return below that of the expected return of illiquid wealth, and as a result, the share ξ tends to grow on average. The consumption share and the illiquid asset share both exhibit fat left tails as a reaction to the fact that ξ tends to drift above optimal levels, which often requires consumption and risky allocations to be reduced.

Figure 6: Holdings and Cost Sensitivity to Return and Correlation



This figure shows the effect varying correlation and the expected return on the second asset on the liquidity cost and optimal allocation of the portfolio.

4.2.1 Parameter Sensitivity

Figure 6 explores how the liquidity cost and base allocations to the illiquid risky assets change as the correlation and respectively the expected return in the illiquid asset increases.

I can see in chart (6a) that for high expected waiting time between trades (the dotted line represents a ten-year friction) the liquidity cost is larger for negative and low correlation levels between the two risky assets. If the two assets were fully liquid, with low correlation, agents could continuously adjust allocations to hedge risk in one of the assets with risk in the other asset. The higher the waiting time between the assets, however, the lower the ability of agents to do so.

The higher the expected return of the second risky asset, the wider the gap between its allocation in the completely liquid case and in the illiquid case (Figure 6d). With higher expected return on the illiquid asset, illiquid wealth tends to grow faster than liquid wealth, so the agent will prefer to transfer larger amounts of wealth when the opportunity arises, and will set the base illiquid allocation to a lower rate.

5 Strategic Allocation with Private Asset Classes

5.1 Model Calibration

To calibrate the model I use data from the capital market assumptions report by JP Morgan (2022). The report provides volatility, expected return, and correlation projections, and is widely used by pension funds and investors to gauge their SAA. The trading probabilities for the illiquid alternative asset classes, on the other hand, are calibrated based on holding period estimates from Ang et al. (2014) and references therein. Tables (2b) and (2c) summarize the overall data.

Table 2b shows the average time between transactions for several private asset classes according to Ang et al. (2014). I will use this as a crude measure of the severity of the asset class illiquidity. Some asset classes, such as direct infrastructure investments, lie on the very extreme of the illiquidity spectrum. For others, such as private equity and hedge funds, secondary markets have only recently come into existence, and even then trading is typically very thin, and funds are priced at large discounts (Kleymenova et al., 2012; Ramadorai, 2012). It is still the norm that private equity and venture capital are exited by finding an appropriate counterparty to buy into the investments once the firms in the fund have matured. From that point of view, the average holding time is an appropriate measure of how easy it is for an investor to rebalance back to the SAAs. In hedge funds, investments are usually redeemed directly from the fund manager once contractual lock-ups and notice periods are satisfied. The contractual clauses for different funds can vary, but I are focused on the SAA, which involves a decision on aggregate investment classes before particular funds are selected to invest in. As a result, the average liquidity properties of hedge funds are appropriate.

I use market projections, rather than actual asset return data for several reasons. First, the SAA decision is by definition forward-looking, and JPM's projections represent a reliable source for long-term investors' expectations about market developments. Second, by using projections rather than directly historical data, I avoid engaging in data biases endemic to the historical returns of many alternative assets. Since markets for these assets are frail, data is often based on appraisals rather than market trading. Often, alternative fund managers have the discretion when and how to report historical returns, with the clear incentive to report returns when they are good. Even when trading occurs, selection bias is not excluded, as the tendency exists for the market to generate observable returns when asset prices are high and sellers are willing to enter the market. Furthermore, empirically, it has been observed that infrequent trading tends to over-smooth historical return data (Ang, 2014), biasing the asset variance downward.

5.2 Quantitative Results

Now, the long-term investor is allowed to allocate wealth between a money market (MM) account that pays the risk-free rate and

1. long-term government bonds (GB) subject to interest rate risk,
2. public equity (PuE)
3. GB and PuE
4. GB, PuE and hedge funds (HF)

Table 2: Model Parameters

(a) Baseline parametrization

Parameter	Definition	Default Value
r	risk-free rate	0.02
β	personal discount rate	0.03
μ_1	first (liquid) risky asset expected return	0.055
σ_1	first (liquid) risky asset volatility	0.14
μ_2	second (illiquid) risky asset expected return	0.055
σ_2	second (illiquid) risky asset volatility	0.14
ρ	correlation	0
$1/\eta$	average time (in years) between a trading opportunity arises	10
γ	risk aversion parameter in the CRRA utility function	6

Note. The table shows a short summary of the parameter values that are used for the baseline parametrization. Unless stated otherwise, these parameters are used for the numerical work later on.

(b) Asset Classes

	JPM Class	$1/\eta$	p	μ	σ
MM	U.S. Cash	0.0	1.00	1.30	0.00
GB	U.S. Long Treasuries	0.0	1.00	2.44	14.00
PuE	AC World Equity	0.0	1.00	6.17	12.75
HF	Diversified Hedge Funds	1.0	0.63	3.82	6.84
Pr	Private Equity	4.0	0.22	9.66	18.68
RE	U.S. Core Real Estate	9.0	0.11	3.82	6.84
Inf	Global Core Infrastructure	55.0	0.02	6.64	10.74

Note. This table shows the expected return and volatility asset class data used to calibrate each of the consequent strategic allocation cases.

(c) Asset Classes, Correlations

	GB	PuE	HF	PrE	RE	Inf
GB	1.0	-0.3	-0.33	-0.57	-0.3	-0.34
PuE	-0.3	1.0	0.76	0.84	0.4	0.6
HF	-0.33	0.76	1.0	0.80	0.37	0.46
PrE	-0.57	0.84	0.80	1.0	0.43	0.65
RE	-0.3	0.4	0.37	0.43	1.0	0.38
Inf	-0.34	0.6	0.46	0..65	0.38	1.0

Note. This table shows the assumed expected correlations between the risky asset classes.

5. GB, PuE and private equity (PrE)

6. GB, PuE and direct real estate holdings (RE)

7. GB, PuE and infrastructure (Inf) investments

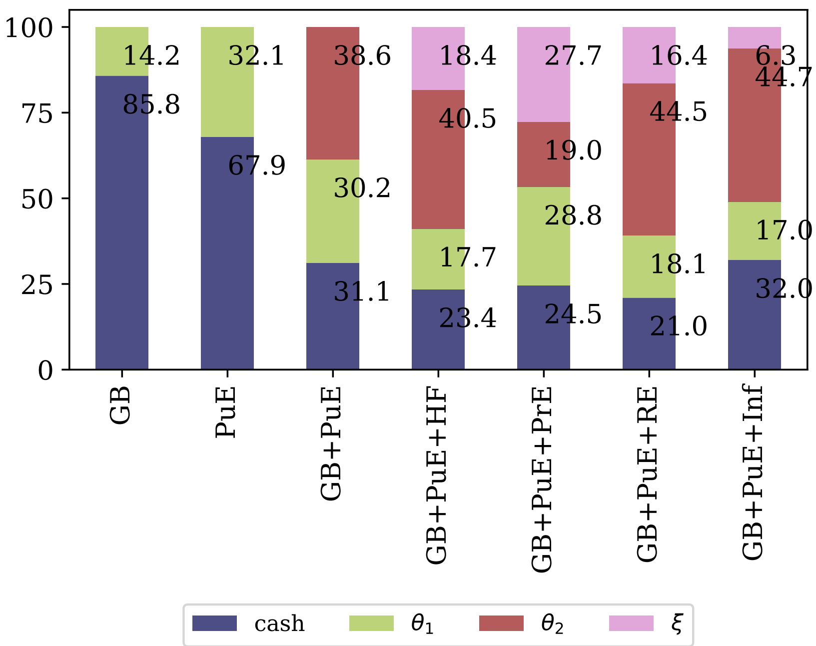
The first three cases can be solved with the continuous rebalancing model of Section 3.2. Each of the cases involving a private asset class is solved through the model incorporating a liquidity friction from Section 3.3.

First, I examine if the introduction of one of the alternative asset class leads to a significant improvement in the CEC of agents following the optimal trading strategy. I evaluate the starting period CEC at the rebalancing point ξ^*, assuming that strategic allocations are set at the moment when agents can transact in the illiquid asset class and will set the illiquid asset allocation at the optimal target. Table 3 shows the resulting percentage improvement in CEC relative to the benchmark continuous rebalancing case (GB+PuE).

Figure 7 shows the allocation to all assets in the portfolio for each of the considered cases. It can be seen that each alternative asset class provides an economically significant improvement in the risk-free equivalent consumption. Most favorable, with 35% improvement, is the combination of traditional asset classes and direct real estate with 16.4% portfolio allocation. Next in the ranking is the combination with infrastructure investments, despite its relatively low allocation of 6.3% and despite the long assumed average waiting time to transact of 55 years.

We can observe that the illiquid model generates diversified portfolios, thus beating down a major concern that theoretical allocation models may generate overly concentrated portfolios. Between 6% and 27% are allocated to the private asset classes when they are available, and in each case, it is optimal to invest in every available asset class.

Figure 7: Strategic Asset Allocation



Note. This figure shows the allocations to alternative asset classes when the ivestor has access to different allocation universes.

Figure 2c examines the shape of the certainty equivalent as a function of the illiquid asset holdings. The more concave the curve around the rebalancing point at ξ^*, the faster the CEC will fall once the illiquid share of wealth floats away from the optimal strategic allocation. The dashed line shows the threshold below which the illiquid opportunity becomes less favorable in CEC terms compared to the benchmark case of investing in bonds and public equity (the *GB+PuE* case). In all cases, I can see that even though the CEC is falling once the illiquid asset share moves away from the optimal ξ^* at the top of the curve, there is still a sizable buffer until reaching the threshold benchmark line. For example, for the most severe illiquid case of investing in infrastructure (Figure 8d), the optimal share of infrastructure is 6.3%. Still, if the share of the illiquid asset stays below around 40% in periods when liquidity is not available, the CEC of this portfolio is still better than the purely liquid portfolio benchmark.

Table 3: Welfare Improvement Due with a Private Asset Class

	$CEC(\xi^*)$	Improvement
GB	1.47	-40%
PuE	2.11	-13%
GB+PuE	2.42	0%
GB+Pu+HF	3.15	30%
GB+Pu+PrE	3.05	26%
GB+Pu+RE	3.27	35%
GB+Pu+Inf	3.17	31%

Note. This Table shows CEC at the rebalancing point. The last column shows the improvement in welfare, measured by the percentage change in CEC relative to the case where investors have access to government bonds and public equity (GB+Pu).

6 Conclusion and Further Research

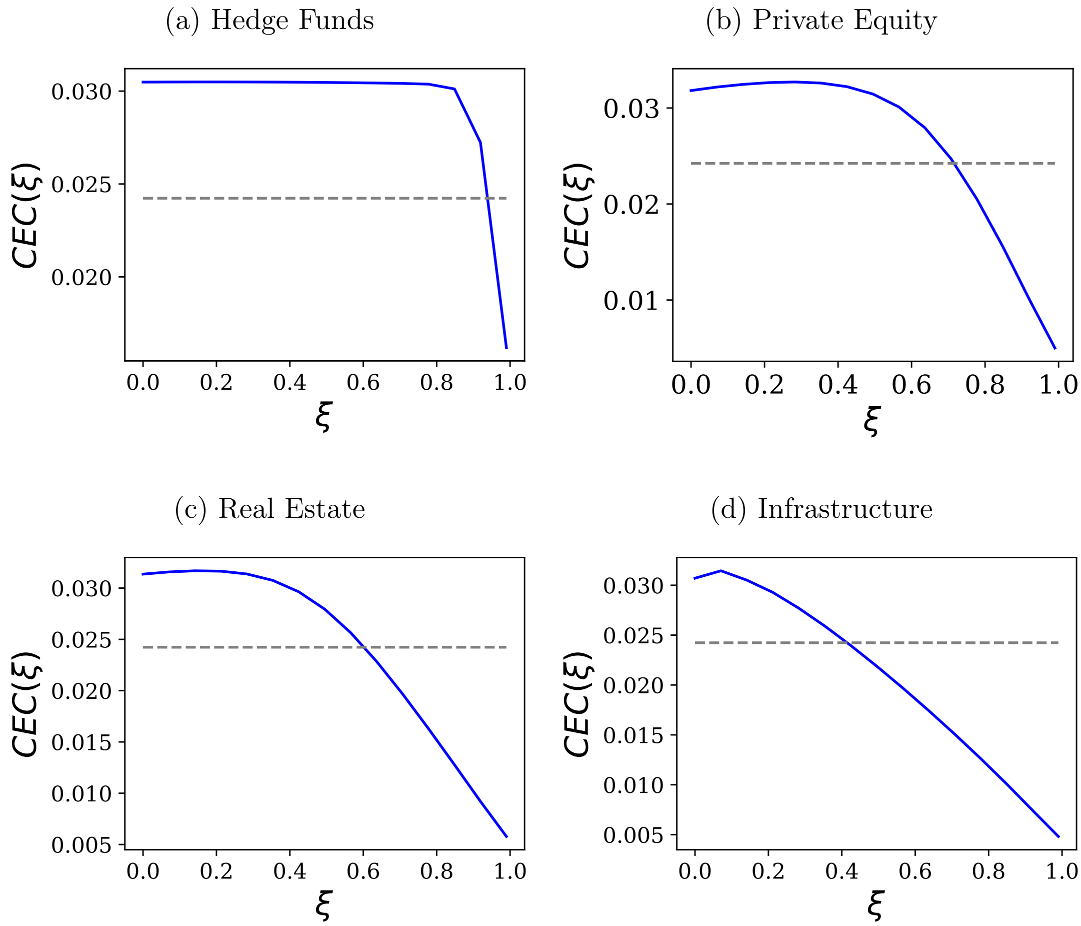
I find economically significant effects from the presence of illiquidity, most notably on the allocation to the illiquid asset. Higher illiquidity results in lower consumption rates, reduced strategic allocation to the illiquid asset, high utility costs, and sub-optimal holding of the liquid asset compared to a market with full liquidity. Yet, balancing the risks of illiquidity against the added benefits of diversification and risk premia, I find that a dynamically optimizing investor can still improve her *ex-ante* welfare relative to a traditional asset mix of government bonds and equity.

In the model, the investor can consume directly only out of the liquid asset, and transfers between liquid and illiquid wealth are allowed with uncertain timing. This gives rise to risks for the investor of not being able to meet consumption objectives as part of the total investment wealth may stay locked up in illiquid holdings.

As the illiquid asset cannot be traded for periods of random length, it exposes the investor to risks that are not hedgeable. The investor cannot always adjust allocations to account for optimality. Thus when market conditions change, the allocation between liquid and illiquid wealth is driven away from optimality.

Calibrating the model to market projections used by long-term investors, I find that despite their illiquid nature, private asset classes have the potential to significantly in-

Figure 8: CEC Profile per Asset Class



Note. This figure shows the curvature of the CEC profile as a function of the illiquid asset share for each alternative class scenario. The gray dashed lines represent the CEC of the liquid investment scenario of holding MM, GB and PuE.

crease the welfare of investors.

The portfolio choice model presented here provides a way for long-term investors to handle illiquid investments in their optimal allocation problem. Yet, further extensions are possible and may be necessary to capture the nuances of each specific asset class. Fixed costs of investment, for example, are a main concern for smaller investors, and may be a reason why smaller pension funds tend to shy away from private assets. Often private assets require particular expertise and organization of the due diligence process that makes such investments an unpopular choice. Furthermore, more research can be put into modeling the common factors that affect returns across asset classes. Extensions of the liquidity friction could, for example, be relaxed by adding the possibility of accessing secondary markets, which are still frail for private assets but have started developing in recent years. Yet, the model presented here provides a sound basis for refinements.

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A Proofs and Derivations

A.1 Liquid Market: The HJB Equation and Optimal Solution

Reexamining the value function (5) over a short time period Δt we can split today's value function into an optimizing decision over the upcoming time span (from t until $t + \Delta t$) and an optimal strategy afterward:

$$V(W_t) = \sup_{(\boldsymbol{\pi}_s, C_s)} \int_t^{t+\Delta t} \mathrm{e}^{-\beta(s-t)} u(C_s) ds + \mathrm{e}^{-\beta \Delta t} E[V(t + \Delta t, W_{t+\Delta t})] \tag{24}$$

Note that the time subscript from the expectation has been dropped, as the wealth dynamics are driven by independent random shocks, implying that knowledge of the present does not help in forecasting the expected value function, so the unconditional expectation can be used instead of the conditional.

We will show that in the limit for small Δt this converges to the continuous time Bellman equation.

Assuming that the strategies for $\boldsymbol{\pi}_s$ and C_s are set and stay constant for the time interval $s \in [t, t + \Delta t)$ where $\Delta t \to 0$, we can write

$$V(t, W_t) = \sup_{(\boldsymbol{\pi}_s, C_s)} \int_t^{t+\Delta t} \mathrm{e}^{-\beta(s-t)} u(C_s) ds + \mathrm{e}^{-\beta \Delta t} E\Big[V(t + \Delta t, W_{t+\Delta t})\Big]$$

Multiplying both sides by $\frac{1}{\Delta t} \mathrm{e}^{\beta \Delta t}$ and rearranging:

$$\frac{\mathrm{e}^{\beta \Delta t} - 1}{\Delta t} V(t, W_t) = \sup_{(\boldsymbol{\pi}_s, C_s)} \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{e}^{-\beta(s-t-\Delta t)} u(C_s) ds + \frac{1}{\Delta t} E\Big[V(t+\Delta t, W_{t+\Delta t}) - V(t, W_t)\Big]$$

Evaluating the above equation for $\Delta t \to 0$, applying the L'Hopital rule, the fact that $\frac{1}{\Delta t} \int_t^{t+\Delta t} f(s) ds = f(t)$ and using the definition of a drift term in a stochastic differential equation by denoting it as $E[dV(t, W_t]$ we get the continuous time Bellman equation as

$$\beta V(t, W_t) = \sup_{(\boldsymbol{\pi}_t, C_t)} u(C_t) + E\Big[dV(t, W_t)\Big] \tag{25}$$

An appealing intuition can be derived by rewriting the Bellman equation (25) in the following form, which follows by reversing the terms in

$$d(\mathrm{e}^{-\beta t} V(W_t)) = -\beta \mathrm{e}^{-\beta t} V(W_t) + \mathrm{e}^{-\beta t} dV(W_t)$$

and substituting in (25) indicates that over any short interval of time the optimal allocation and consumption strategies have to ensure that the discounted utility exactly offsets any expected decline in the discounted value function:

$$\sup_{(\boldsymbol{\pi}_t, C_t)} \mathrm{e}^{-\beta t} u(C_t) + E\left[d(\mathrm{e}^{-\beta t} V(W_t))\right] dt = 0 \tag{26}$$

Apply the Itô rule on the stochastic term $dV(t, W_t)$ in (25) and substitute in the budget constraint for dW_t

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + V_W dW_t + 1/2 V_{WW} [dW_t]^2 \\ &= \frac{\partial V}{\partial t} dt - C_t V_W dt + V_W W_t (r dt + \boldsymbol{\pi}_t' (\boldsymbol{\mu} - r\mathbb{1}) dt + \boldsymbol{\pi}_t' \boldsymbol{\sigma} \boldsymbol{dZ}_t) + \frac{1}{2} V_{WW} W^2 \boldsymbol{\pi}_t' \boldsymbol{\Sigma} \boldsymbol{\pi}_t dt \end{aligned}$$

Then use the fact that $E[dZ_j] = 0$, and that in the infinite horizon case $\frac{\partial V}{\partial t} = 0$. Consequently we can derive the drift term of the Bellman equation as

$$E\left[dV(t, W_t)\right] dt = \left[-C_t V_W + V_W W_t(r + \boldsymbol{\pi}_t'(\boldsymbol{\mu} - r\mathbb{1})) + \frac{1}{2} V_{ww} W^2 \boldsymbol{\pi}_t' \boldsymbol{\Sigma} \boldsymbol{\pi}_t\right] dt$$

which after substitution yields the HJB equation of Proposition (27):

$$\beta V(t, W_t) = \sup_{(\boldsymbol{\pi}_t, C_t)} u(C_t) - C_t V_W + V_W W_t(r + \boldsymbol{\pi}_t'(\boldsymbol{\mu} - r\mathbb{1})) + \frac{1}{2} V_{ww} W^2 \boldsymbol{\pi}_t' \boldsymbol{\Sigma} \boldsymbol{\pi}_t$$

We are using the verification approach for solving the HJB equation. This involves first solving the optimization part and finding the optimal $\boldsymbol{\pi}$ and C with the yet unknown value function, as was done in equations (**??**) and (7). As a second step, we solve the resulting HJB equation (in our case making a guess for the value function). It not always possible or easy, however, to solve the HJB explicitly as we did here. In some cases, we need to use numerical procedures to do so. These methods are discussed in detail in Section **??**

Substituting the optimal consumption allocation terms of equations (**??**) and (7) in the HJB equation of (6) simplifies it to:

$$\beta V = \frac{\gamma}{1-\gamma} V_W^{1-1/\gamma} + r W_t V_W - \frac{1}{2} \frac{V_W^2}{V_{WW}} \|\boldsymbol{\lambda}\|^2 \tag{27}$$

This second order Partial Differential Equation can be solved by making a guess for the value function of the form

$$V(W_t, t) = g(t)^\gamma \frac{W_t^{1-\gamma}}{1-\gamma}$$

Substituting in (27) verifies that the guess solves the PDE for $g(t) = \frac{1}{A}$, where

$$\begin{aligned} A &\equiv \frac{\beta + r(\gamma - 1)}{\gamma} + \frac{1}{2} \frac{\gamma - 1}{\gamma^2} \|\boldsymbol{\lambda}\|^2 \\ &= \frac{\beta + r(\gamma - 1)}{\gamma} + \frac{1}{2} \frac{\gamma - 1}{\gamma^2} (\boldsymbol{\mu} - r\mathbf{1})'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \end{aligned} \tag{28}$$

The optimal consumption share is thus $c = \frac{C_t}{W_t} = A$. Note that $g(t)$ is a constant function.[11]

A.2 Illiquid Market: Homogeneity of the Value function

The proof follows from Ang et al. (2014). For CRRA utility in particular, the value function $V(W_t, X_t)$ is homogeneous of degree $1-\gamma$, i.e. $V(kW_t, kX_t) = k^{1-\gamma} V(W_t, X_t)$ for any $k > 0$. This is a direct consequence of the fact that the budget constraint dynamics are linear in wealth and have constant moments (independent of the corresponding wealth states). Then it is reasonable to accept that for an optimal solution $\{W_s^*, X_s^*, dI_s^*, c_s^*, \theta_s^*\}$ also $\{kW_s^*, kX_s^*, kdI_s^*, c_s^*, \theta_s^*\}$ will be optimal as well for any $k > 0$, so that scaling both

[11] In the finite horizon case, it will be a deterministic function of time, introduce time dependency in the value function.

liquid and illiquid wealth up or down by the same number does not change the optimal investment and consumption rates given that we also scale the wealth transfers dI by the same number.

Then we can write

$$V(kW_t, kX_t) = \sup_{\theta, dI, c} E_t \left[\int_t^\infty \mathrm{e}^{-\beta(s-t)} \frac{(kc_s(1-\xi_s)W_s)^{1-\gamma}}{1-\gamma} ds \right] = k^{1-\gamma} V(W_t, X_t)$$

As a result

$$V(W_t, X_t) = \left(\frac{1}{k}\right)^{1-\gamma} V(kW_t, kX_t)$$

Setting $k = 1/(W_t + X_t)$ we get

$$V(W_t, X_t) = (X_t + W_t)^{1-\gamma} V((1-\xi), \xi) = (X_t + W_t)^{1-\gamma} H(\xi)$$

where ξ is the portion of total wealth invested in the illiquid asset x and $(1-\xi)$ is the portion invested in the liquid asset. The additional proof that $H(\xi)$ is concave is available in (Ang et al., 2014).

A.3 Illiquid Market: HJB Equation

Here we derive the equation (14). Starting with the continous time Bellman equation (25) with $\xi_t = \frac{X_t}{W_t + X_t}$ as the proportion of total wealth invested in the illiquid asset and θ_t as the proportion of liquid wealth invested in the illiquid asset we get:

$$\begin{aligned}
\beta V(X_t, W_t) &= \sup_{(\xi_t, \theta_t, C_t)} \{u(C_t) + E[dV(X_t, W_t)]\} \\
&= \sup_{(\xi_t, \theta_t, C_t)} \{u(C_t) + V_W W_t (r + (\mu_1 - r)\theta_t) - V_W C_t + V_X X_t \mu_2 \\
&+ \frac{1}{2} V_{WW} W_t^2 \theta^2 \sigma_1^2 + \frac{1}{2} V_{XX} X^2 \sigma_2^2 + V_{WX} W_t X_t \sigma_2 \sigma_1 \rho \theta + \eta(V^* - V(W_t, X_t))\}
\end{aligned}$$

where in the last line the Ito rule for jump processes (see (Shreve, 2004, Chapter 11)) is applied such that:

$$\begin{aligned}
E[dV] = E &\left[V_w dW^c + V_X dX^c + \frac{1}{2}\left(V_{WW}[dW^c]^2 + V_{XX}[dX^c]^2 + 2V_{WX}[dX^c dW^c]\right) + (V^* - V)dN \right] \\
&= (r + \theta_t(\mu_1 - r\mathbb{1}))\, V_W W_t + C_t V_W + \frac{1}{2} V_{WW} W_t^2 \theta_t^2 \sigma_1^2 \\
&+ V_X \mu_2 + \frac{1}{2} V_{XX} X^2 \sigma_2^2 + V_{WX} W X \theta \sigma_2 \sigma_1 \rho + \eta(V^* - V(W_t, X_t))
\end{aligned}$$

where we denote as dX^c and dW^c the continuous portion of the budget constraints and thus get the corresponding quadratic variation terms:

$$\begin{aligned}
dW_t^c &= (r + (\mu_1 - r)\theta_t) W dt - C_t + \theta_t \sigma_1 W dZ_{1t} \\
dX_t^c &= \mu_2 X_t dt + \sigma_2 \rho X_t dZ_{1t} + \sigma_2 \sqrt{1-\rho^2} X_t dZ_{2t} \\
[dW_t^c]^2 &= \theta_t^2 \sigma_1^2 W^2 dt \\
[dX_t^c]^2 &= \sigma_2^2 X_t^2 \\
[dW_t^c dX_t^c] &= \theta \sigma_2 \sigma_1 \rho W X
\end{aligned}$$

and denoting the jump size in the value function when a liquidity opportunity arises as $V^* - V(W_t, X_t)$, such that the expected jump size over a short period of time is $\eta(V^* - V(W_t, X_t))dt$. Note that whenever the Poisson jump process hits, we have inferred that the value function jumps to $V^* = (W_t + X_t)^{1-\gamma} H^*$.

This yields the given HJB equation

$$\mathcal{L}^C + \mathcal{L}^\theta + \mathcal{L} - \beta V(W_t, X_t) = 0$$

where

$$\begin{aligned}
\mathcal{L}^C &= \sup_{C_t} \left\{ u(C_t) - C_t V_W \right\} \\
\mathcal{L}^\theta &= \sup_{\theta_t} \left\{ (r + \theta_t(\mu_1 - r\mathbb{1}))\, V_W W_t + \frac{1}{2} V_{WW} W_t^2 \theta_t^2 \sigma_1^2 + V_{WX} W X \theta_t \sigma_2 \sigma_1 \rho \right\} \\
\mathcal{L} &= V_X \mu_2 + \frac{1}{2} V_{XX} X^2 + \eta(V^* - V(W_t, X_t))
\end{aligned}$$

To solve for consumption and liquid investment in terms the illiquid asset holdings, we apply the substitutions

$$\begin{aligned}
\xi_t &= \frac{X_t}{W_t + X_t} \\
V(W_t, X_t) &= (W_t + X_t)^{1-\gamma} H(\xi_t)
\end{aligned}$$

such that

$$\begin{aligned}
V_W &= (W + X)^{-\gamma} \left((1-\gamma)H(\xi) - \xi H'(\xi)\right) \\
V_X &= (W + X)^{-\gamma} \left((1-\gamma)H(\xi) + (1-\xi)H'(\xi)\right) \\
V_{WW} &= (W + X)^{-\gamma-1} \left(-\gamma(1-\gamma)H(\xi) + 2\xi\gamma H'(\xi) + \xi^2 H''(\xi)\right) \\
V_{XX} &= (W + X)^{-\gamma-1} \left(-\gamma(1-\gamma)H(\xi) - 2(1-\xi)\gamma H'(\xi) + (1-\xi)^2 H''(\xi)\right) \\
V_{WX} &= (W + X)^{-\gamma-1} \left(-\gamma(1-\gamma)H(\xi) - 2(1-\xi)\gamma H'(\xi) - (1-\xi)H''(\xi)\xi\right)
\end{aligned}$$

The partial derivatives are implied using the Chain Rule such that $\frac{\partial \xi}{\partial X} = \frac{1-\xi}{W+X}$ and $\frac{\partial \xi}{\partial W} = -\frac{\xi}{W+X}$, where $H'(\xi_t)$ and $H''(\xi_t)$ denote the first and second partial derivatives of $H(\xi)$ with respect to ξ.

Applying that in the HJB equation we can solve for optimal consumption

$$c_t^* = \left((1-\gamma)H(\xi_t) - H'(\xi_t)\xi_t\right)^{-\frac{1}{\gamma}} (1-\xi_t)^{-1}$$

and optimal liquid risk asset investment

$$\theta_t^* = -\frac{k_1 H(\xi_t) + k_2 H'(\xi_t) + k_3 H''(\xi_t)}{k_4 H(\xi_t) + k_5 H'(\xi_t) + k_6 H''(\xi_t)}$$

where $k_1, .., k_6$ are known constants defined by the market parameters and the agent's risk aversion such that

$$\begin{aligned}
k_1 &= -(1-\gamma)(\mu_1 - r) + \gamma(1-\gamma)\sigma_2 \rho \sigma_1 \xi \\
k_2 &= (\mu_1 - r)\xi - \sigma_2 \sigma_1 \rho \xi \gamma (2\xi - 1) \\
k_3 &= -\sigma_2 \sigma_1 \rho \xi^2 (1-\xi) \\
k_4 &= -\gamma(1-\gamma)(1-\xi)\sigma_1^2 \\
k_5 &= 2\gamma\xi(1-\xi)\sigma_1^2 \\
k_6 &= \xi^2(1-\xi)\sigma_1^2
\end{aligned}$$

B Discretization & Numerical Solution Methods

B.1 Discretization and Numerical Approaches

The discretized version of the Bellman Equation for the problem (11 - 12), with $\delta = e^{-\beta \Delta t}$ as the discrete-time discount factor, can be written as [12]:

$$V(W_t, X_t) = \max_{(\theta_t, dI_t, c_t \in \mathcal{A})} \{u(C_t)\Delta t + \delta E_{W_t, X_t}[V(W_{t+\Delta t}, X_{t+\Delta t})]\} \tag{29}$$

To get some intuition on the discretization step, note that the discrete Bellman equation

$$V(W_t) = \sup_{(\pi_t, C_t)} \{u(C_t)\Delta t + e^{-\beta \Delta t} E[V(W_{t+\Delta t})]\}$$

can be shown to converge to its continuous counterpart for $\Delta t \to 0$. Multiply the equation by $e^{\beta \Delta t}$, subtract $V(W)$ from both sides and divide by Δt and this results in:

$$\frac{e^{\beta \Delta t} - 1}{\Delta t} V(W) = \sup_{(\pi_t \in \mathcal{R}^d, c_t \geq 0)} \{u(c_t) + \frac{1}{\Delta t} E_t[V(W_{t+\Delta t}) - V(W_t)]\}$$

Let $\Delta t \to 0$, then $e^{\beta \Delta t} \to 1$ and also by the L'Hôpital rule we have that $\frac{e^{\beta \Delta t} - 1}{\Delta t} \to \beta$. As a result $\frac{1}{\Delta t} E_t[V(W_{t+\Delta t}) - V(W_t)] \to E_t[dV]$ which makes discretized equation equivalent to the continuous time version of (26).

The expectation is conditional on liquid wealth W_t and illiquid wealth X_t. Conditioning on t is not needed as we are looking at an infinite horizon problem and both wealth processes have the Markovian property. Using the homothetic properties of the CRRA utility function we can write

$$\begin{aligned} V(W_t, X_t) &= Q_t^{(1-\gamma)} H(\xi_t) \\ u(C_t) &= u(c_t W_t) \\ &= Q_t^{1-\gamma} u(c_t(1 - \xi_t)) \end{aligned}$$

where $Q_t = W_t + X_t$ stands for total wealth and $W_t = (1 - \xi_t)Q_t$.

Knowing the function $H(\xi)$ allows us to find the optimal ratio of illiquid to total wealth ξ^* as the maximizing value for that function. Our goal is then to write the Bellman equation and to solve for the control variables in terms of ξ_t and $H(\xi_t)$.

The Bellman Equation can then also be written as:

$$\begin{aligned} V(Q_t, \xi_t) &= \max_{(\theta_t, \xi_t, c_t \in \mathcal{R})} \{u(c_t(1 - \xi_t)Q_t)\Delta t + \delta E_{Q_t, \xi_t}[V(Q_{t+\Delta t}, \xi_{t+\Delta t})]\} \\ Q_t^{(1-\gamma)} H(\xi_t) &= \max_{(\theta_t, \xi_t, c_t \in \mathcal{R})} \{Q_t^{(1-\gamma)} u(c_t(1 - \xi_t))\Delta t + \delta E_{Q_t, \xi_t}[Q_{t+\Delta t}^{(1-\gamma)} H(\xi_{t+\Delta t})]\} \end{aligned} \tag{30}$$

As illustrated on Figure 2, with probability p the agent will be able to trade the illiquid asset next period and will bring the portion of illiquid wealth to the desired optimal level $\xi^* = arg \max_\xi H(\xi)$. This is done by setting the transfer dI_t between liquid and illiquid wealth to accommodate optimal asset allocations, so:

$$dI_t = \begin{cases} \xi^* Q_t - X_{t-} & \text{with probab. } p \\ 0 & \text{with probab. } 1 - p \end{cases}$$

where X_{t-} is the level of illiquid holdings just before rebalancing takes place. As a result, we eliminated the need to solve directly for dI_t by taking into account that the agent will set $\xi_{t+\Delta t} = \xi^*$ whenever trading is possible. If trading is not possible, the agent cannot rebalance and is stuck with sub-optimal levels of illiquid holdings $\xi_{t+\Delta t} = X_{t+\Delta t}/Q_{t+\Delta t}$, and the ratio will float away from last period's value as the prices of the two risky assets move erratically.

Combining those two states with the corresponding realization probabilities, we can use the law of iterated expectations[13] to drop the conditional expectation with respect to ξ.[14]

$$Q_t^{(1-\gamma)} H(\xi_t) =$$
$$\max_{(\theta_t, dI_t, c_t \in \mathcal{R})} \left\{ Q_t^{(1-\gamma)} u(c_t(1-\xi_t))\Delta t + \delta \Big(pE_{Q_t}[Q_{t+\Delta t}^{(1-\gamma)}] H^* + (1-p) E_{Q_t}[Q_{t+\Delta t}^{(1-\gamma)} H(\xi_{t+\Delta t})] \Big) \right\}$$

Canceling out wealth from both sides of the equation yields desired form of the equation. Note, that by canceling out Q_t and by embedding it in the expectations the conditional part of the expectation with respect to Q has been dropped:

$$\begin{aligned} H(\xi_t) = & \\ & \max_{(\theta_t, \xi_t, c_t)} \left\{ u(c_t(1-\xi_t))\Delta t + \delta \left(pH^* E_{\xi_t} \left[R_{q,t+\Delta t}^{1-\gamma} \right] \right. \right. \\ & \left. \left. + (1-p) E_{\xi_t} \left[R_{q,t+\Delta t}^{1-\gamma} H(\xi_{t+\Delta t}) \right] \right) \right\} \end{aligned} \tag{31}$$

where $R_{q,t+\Delta t}$ is the growth of total wealth in the next period net of current consumption, and $\xi_{t+\Delta t}$ is the holdings ratio given that rebalancing is not possible. The laws of motion then for these two variables are such that:

$$\begin{aligned} R_{q,t+\Delta t} &= \frac{Q_{t+\Delta t}}{Q_t} = \frac{W_t R_{w,t+\Delta t} + X_t R_{x,t+\Delta t}}{Q_t} \\ &= (1-\xi_t) R_{w,t+\Delta t} + \xi R_{x,t+\Delta t} \\ \xi_{t+\Delta t} &= \frac{X_{t+\Delta t}}{Q_{t+1}} = \frac{X_t}{Q_t} \frac{X_{t+\Delta t}}{X_t} \frac{Q_t}{Q_{t+\Delta t}} \\ &= \xi_t \frac{R_{x,t+\Delta t}}{R_{q,t+\Delta t}} \end{aligned} \tag{32}$$

Where $R_{w,t+\Delta t}$ and $R_{x,t+\Delta t}$ are the discretized gross returns on liquid and illiquid wealth respectively after factoring out consumption. Using Euler time-discretization of the continuous stochastic process we get:

$$\begin{aligned} R_{w,t+\Delta t} &= \frac{W_{t+\Delta t}}{W_t} = (r + (\mu_1 - r)\theta_t - c_t)\Delta t + \sigma_1 \sqrt{\Delta t} \Delta Z_{1,t} \\ 1 + R_{x,t+\Delta t} &= \frac{X_{t+\Delta t}}{X_t} = \mu_2 \Delta t + \sigma_2 \rho \Delta Z_{1,t} + \sigma_2 \sqrt{1-\rho^2} \sqrt{\Delta t} \Delta Z_{2,t} \end{aligned} \tag{33}$$

[13]The law as applied here states that $E(X|Y_1) = E(E(X|Y_1, Y_2)|Y_2)$.

[14]A similar approach to decomposing the Bellman equation is used also by Moore and Young (2006). They apply it as part of a Markov Chain approximation method in order to solve a portfolio choice problem with insurable loss which occurs with a Poisson probability.

where $\Delta Z_{1,t}$ and $\Delta Z_{2,t}$ are two independent standard normal random variables. We use a Gaussian quadrature approach for the space-discretization of the normal distribution and for the evaluation of the expectations in the Euler equation.

The discretized Bellman equation can then be solved through value function iteration combined with standard numerical techniques. In Appendix B.2 we show the algorithm used for the purpose.

B.2 Value Function Iteration

Based on the discretization from Section 3.3.2, we can solve the portfolio choice dynamic problem through value function iteration. Discretizing ξ and simulating the system one period ahead will allow us to iterate (31) until the iterative approximation of the value function $H(\xi)$ converges. The procedure will eventually yield a numerical approximation of the value function and the optimal solution for the policy functions $c(\xi)$ and $\theta(\xi)$. So, the goal is to find the vector $\tilde{h} = [h_1, h_2, ..., h_N]$ on a grid $\tilde{\xi}$ which best approximates the function $H(\xi)$. We use the following iterative procedure:

Initialization: Discretize the random space (the random variables ΔZ_1 and ΔZ_2 in (33)) through a simulation or strategically selected (Gaussian) quadrature points. Select an approximation grid of N gridpoints for ξ_t: $\tilde{\xi} = \{\xi_1, ..., \xi_N\} \in [0; 1)$, $j = 1, .., N$. We use $N = 20$. Select a class of approximating functions $h(\boldsymbol{a}; \tilde{\xi})$ which will approximate the true value function $H(\xi)$ and initialize the functional parameters $\boldsymbol{a}$. Select an initial maximum of the value function h^*.[15] As a starting point for the iteration we use the analytical two-asset Merton solution. We can then initiate the following iterative algorithm.

1. **Optimization:** For each ξ_j in the grid compute optimal consumption and liquid asset allocation:

$$\begin{aligned} c^{*,k}, \theta^{*,k+1} = & \\ & \arg\min \left\{ u(c(1-\xi_j))\Delta t + \delta \left(p h^* E\left[\hat{q}(c, \theta, \xi_j)^{1-\gamma} \right] \right. \right. \\ & \left. \left. + (1-p) E\left[h\left(\boldsymbol{a}^k; \hat{\xi}(c, \theta, \xi_j) \right) \hat{q}(c, \theta, \xi_j)^{1-\gamma} \right] \right. \end{aligned}$$

   where $\hat{\xi}()$ and $\hat{q}()$ are next-period's dynamics calculated through (32), and $k = 1, ..., p$ is a counter measuring the iteration run.

2. **Update:** For each ξ_j and the optimal control policies found in the previous step update values of the value function that lie on the grid. Update the fit of the approximation function $h(\boldsymbol{a}^{k+1}; \xi)$ based on the new values. Update $h^{*,k+1} = \arg\max_\xi h(\boldsymbol{a}^{k+1}; \xi)$.

3. **Stopping:** The algorithms stops if $\left\| \ln(h(\boldsymbol{a}^{k+1}; \xi)) - \ln(h(\boldsymbol{a}^k; \xi)) \right\|^2 < \epsilon$, otherwise we go to **Step 1**. I use $\epsilon = 0.1^6$

The expectation operator in the second step is evaluated through multi-variate quadrature (Judd, 1998; Cai et al., 2013; Cai and Judd, 2014).

[15] We have successfully tried simple polynomials and cubic splines for the purpose. Eventually, the later turned out to provide more flexibility, so all the shown results are derived through this functional form.