Moment Conditions and Time-Varying Risk Premia

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Abstract

This paper proposes a novel approach for estimating linear factor pricing models with dynamic risk premia based on a generalized method of moments framework. Time-varying risk prices and exposures follow an updating scheme that aims for the steepest descent of the conditional moment-criterion function at time t. The most informative moment for inferring risk premium dynamics comes from the cross-sectional pricing equation estimated in the second stage of the popular Fama-MacBeth regression approach. Monte Carlo results show that the new approach is able to adequately filter various types of risk premium dynamics. An application to the Fama-French 3-factor model shows that the GMM-based procedure can largely reduce pricing errors compared to other dynamic and static approaches. The results show that risk premia dynamics vary across factors, and while they are generally countercyclical, they exhibit significant declines at the beginning of crisis periods.

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1 Introduction

Financial theories interpret expected returns over a risk-free interest rate, known as excess returns, as compensation for the risk of the investment. Traditional factor asset pricing models (Fama and French, 1993; Carhart, 1997) describe these premia with risk prices (lambdas) demanded by investors for each unit of exposure (beta) to a financial or macroeconomic source of risk. Estimation of these linear factor models, which are widely used in empirical financial research, are typically conducted with the traditional two-step regression approach of Fama and MacBeth (1973) (henceforth denoted with FMB) or Generalized Method of Moments (GMM) frameworks following Hansen (1982). However, despite crucial evidence in the literature that risk premia vary over time (Campbell and Shiller, 1988; Fama and French, 1989; Cochrane, 2011), risk exposures of financial securities, as well as risk prices and thus risk premia are typically assumed to be constant over time in these estimation approaches. Moreover, the recent factor timing literature (Moreira and Muir, 2017; Haddad et al., 2020; Ehsani and Linnainmaa, 2022; Arnott et al., 2023; Neuhierl et al., 2023) has shown that the dynamics of risk premia can explain economically relevant excess returns and are therefore important for understanding the factor structure of financial returns.

This paper proposes a moment-based approach to estimate time-varying risk premia in linear factor pricing models. The approach builds on a small set of asset pricing moments which is often used for GMM-based estimation of unconditional asset pricing models following the methodology of Hansen (1982). We extend this baseline model with an observation-driven updating scheme for risk exposures and prices in order to achieve a conditional factor pricing model. The parameter updating follows the general approach of Creal et al. (2024) to let the dynamics be driven by the influence function of the conditional GMM estimator in each time period. This intuitively provides a steepest descent improvement of the local GMM criterion function in the corresponding time period. It turns out that a such constructed updating mechanism adjusts risk prices according to regression errors from the cross-sectional regression performed in the second stage of the FMB procedure. Thus, instead of finding a risk price that minimizes these errors on average, the procedure here uses these cross-sectional pricing errors to infer risk price dynamics. If the idiosyncratic innovation of an asset moves with a factor innovation, the factor exposure of that asset is increased by raising the corresponding beta to remove the unwanted comovement. Static parameters in the introduced Moment-Based Dynamic Asset Pricing Model (MDAPM) can be readily estimated using an instrumented GMM approach. As usual for GMM, more efficient estimates can be obtained by performing moment minimization with the optimal weighting matrix in a second stage.

We conduct a Monte Carlo study to examine the performance of the MDAPM. In a scenario with realistically low signal-to-noise ratio, the MDAPM can recover various risk premium dynamics such as cycles and structural breaks. Compared to a static benchmark, the new dynamic approach successfully reduces both pricing and risk premium prediction errors.

An empirical application to the Fama-French 3-factor model (Fama and French, 1993) on a cross-section of 25 portfolios sorted by size and value shows that the dynamic GMM procedure can substantially reduce pricing and risk premium prediction errors compared to an unconditional model. Moreover, the pricing errors are even substantially smaller than those obtained with the regression-based dynamic asset pricing model of Adrian et al. (2015), which uses stock return predictors to infer risk price movements. The results also suggest that in the cross-section of the 25 portfolios, the variation in the risk premium is driven more strongly by changes in risk prices than by changes in betas.

We document two main observations with respect to filtered risk premia. First, risk premia initially fall at the beginning of recessions for several months but rise afterwards. A pattern that have also been documented by Gómez-Cram (2022) for the market premium. Thus, it appears that after an initial downward adjustment of expectations, adverse events lead investors to demand higher compensation for risk. This is puzzling given the numerous references to the countercyclical behavior of expected excess returns on stocks, such as Fama and French (1989), Ferson and Harvey (1991), and Lustig and Verdelhan (2012). The second observation is that filtered risk prices are particularly consistent with the trajectories of stock return predictors in the case of the market risk premium, while there are substantial discrepancies for size and value premia. Therefore, predictors that predict overall market returns may not be adequate or sufficient instruments for predicting size and value risk premia.

The search for empirical methods to estimate factor asset pricing models with time-varying risk premia has recently received renewed attention. Regression-based approaches such as Adrian et al. (2015, 2019), Gagliardini et al. (2016, 2020) and Chaieb et al. (2021) use instrument variables to explain the time dynamics of the parameters λ and β. A common drawback of these approaches is that time-varying risk premia can only be identified with respect to a filtration spanned by the set of instruments employed. This is particularly problematic given that the literature on the predictability of returns is still debating whether returns are predictable at all and what the appropriate predictors are. Thus, even if the literature finds that risk premia are significantly time-varying, the interpretation of the filtered premia series may be misleading due to inappropriate or simply missing predictors. The GMM-based dynamic model proposed here avoids this problem by filtering out the dynamics of risk premia from the full set of available assets and factors. The time series predictors used in the regression-based approaches can additionally be used as instruments to identify factor mean dynamics. Umlandt (2023) provides an observation-driven filter to estimate dynamic financial risk premia without requirement to specify instrument variables for the time dynamics. This likelihood-based method follows the generalized autoregressive score approach of Creal et al. (2013). Although, the latter approach circumvents a misspecification bias due to inappropriate time series predictors, it requires explicit distributional assumptions. In contrast, the asset pricing literature often refrains from posing distributional assumptions and specifies a set of moment restrictions instead that are typically derived from no-arbitrage assumptions. The GMM-based approach proposed here develops a filter for risk premia based only on such a set of moment restrictions, but otherwise follows the logic of the likelihood-based filter. Thus, our approach can be seen as an alternative that is robust to distributional misspecification. Another advantage of our approach is that, in contrast to the likelihood-based model, we do not have to explicitly estimate the covariance matrix of the asset-specific innovations, which massively reduces the number of parameters in the numerical optimization and thus lowers the computational burden of the estimation procedure.

The remainder of the paper is organized as follows. Section 2 introduces and discusses the dynamic GMM framework for linear factor pricing models. The empirical application on the Fama-French 3-Factor model is presented in Section 3. Section 4 concludes.

2 Dynamic GMM Model

In the following we introduce the dynamic GMM-model upon a fairly general baseline factor pricing model that in a similar fashion serves as basis for most of the employed empirical methods as, for example, Fama and MacBeth (1973), Ang and Kristensen (2012), Adrian et al. (2015, 2019), Umlandt (2023), and Gagliardini et al. (2016, 2020).

2.1 Baseline Model

Let $r_t = (r_t^1, \ldots, r_t^N)^\top$ denote the N-dimensional vector representing the excess returns of N different assets at time $t \in \{0, \ldots, T\}$. The underlying data-generating process is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\mathcal{F}_t = \sigma(\{r_t, \ldots, r_0\})$ representing the set of information available at time t. Suppose the risk in the economy is described in terms of K_f risk factors covered in the state vector f_t that follows

$$f_t = \phi_0 + \phi z_{t-1} + u_t, \qquad t = 1, \ldots, T, \tag{1}$$

where u_t is an independently and identically distributed zero mean moise term with covariance matrix Σ_u ($u_t \overset{iid}{\sim} (0, \Sigma_u)$), and z_t is a K_z-dimensional vector of lagged predictors adapted to $\mathcal{F}_t$. Predictors may be external as well as past factor observations. We refer to equation (1) as the risk factor model.

Assume the existence of a unique stochastic discount factor (SDF) m_t that prices every asset $i \in \{1, \dots N\}$ according to

$$\mathbb{E}_{t-1}(m_t r_t) = 0, \tag{2}$$

where $\mathbb{E}_{t-1}$ denotes the conditional expectation with respect to time $t-1$ information $\mathcal{F}_{t-1}$. The Euler equation (2) can be used to compute the conditional covariance between the SDF and the asset return as

$$Cov_{t-1}(r_t, m_t) = -\mathbb{E}_t(r_t)\mathbb{E}_t(m_t). \tag{3}$$

Regressing the demeaned returns on the factor innovations u_t yields an N-dimensional idiosyncratic noise term $e_t \overset{iid}{\sim} (0, \Sigma_\varepsilon)$ that is orthogonal to u_t. Taken together with (3), the return can be decomposed as

$$r_t = \mathbb{E}_{t-1}(r_t) + (r_t - \mathbb{E}_{t-1}(r_t)) = -\frac{Cov_{t-1}(r_t, m_t)}{\mathbb{E}_{t-1}(m_t)} + \beta_t u_t + e_t, \tag{4}$$

where $\beta_t = Cov_{t-1}(r_t, u_t)\Sigma_u^{-1}$ denotes the $N \times K$-dimensional matrix of risk exposures.

In order to transform equation (4) into a cross-sectional pricing model, assume the SDF to be affine-linear in the economy's risk factor innovations; that is,

$$\frac{m_{t+1} - \mathbb{E}_{t-1}(m_t)}{\mathbb{E}_{t-1}(m_t)} = -\lambda_t^\top \Sigma_u^{-1} u_t \tag{5}$$

with time-variant price of risk vector λ_t of dimension K. Plugging the SDF into the return decomposition (4) yields a standard beta representation given by

$$r_t = Cov_{t-1}\left(r_t, u_t\right)\Sigma_u^{-1}\lambda_t + \beta_t u_t + e_{i,t} \tag{6}$$

$$= \beta_t\lambda_t + \beta_t u_t + e_t. \tag{7}$$

The return decomposition (7) therefore consists of a predictable risk premium $\beta_t\lambda_t$ that compensates risk exposures, an unpredictable component $\beta_t u_t$ depending on risk factor innovations and an asset-specific innovation term e_t. Representation (7) is also referred to as the cross-sectional pricing equation.

2.2 Moment Conditions

Given constant risk premia, that means $\beta_t \equiv \beta$ and $\lambda_t \equiv \lambda$, the model in Section 2.1 can be estimated with GMM using a concise set of moment conditions. The static model moment conditions introduced here will serve as the basis for the dynamic model that will be derived in the following sections.

The first two sets of moment conditions stem from the factor model equation (1) and are given by

$$\mathbb{E}_{t-1}\left[u_t\right] = \mathbb{E}_{t-1}\left[f_t - \phi_0 + \phi z_{t-1}\right] = 0 \tag{8}$$

$$\mathbb{E}_{t-1}\left[\operatorname{vec}\left(u_t z_{t-1}^\top\right)\right] = \mathbb{E}_{t-1}\left[\operatorname{vec}\left(\left(f_t - \phi_0 + \phi z_{t-1}\right) z_{t-1}^\top\right)\right] = 0 \tag{9}$$

where equation (8) states the factor innovation to be zero on average, whereas equation (9) is a standard orthogonality assumption that requires the set of instruments z_{t-1} to be uncorrelated with the factor innovations u_t.

Furthermore, we use the aforementioned orthogonality assumption between factor innovations and idiosyncratic innovations to set up the third moment condition given by

$$\mathbb{E}_{t-1}\left[\text{vec}\left(e_t u_t^\top\right)\right] = \mathbb{E}_{t-1}\left[\text{vec}\left((r_t - \beta\lambda - \beta u_t)u_t^\top\right)\right] = 0. \tag{10}$$

According to Condition (10), the conditional covariance between the return vector and factor innovations must satisfy $\mathbb{E}_{t-1}\left[r_t u_t^\top\right] = \beta\Sigma_u$. Thus, the third moment condition (10) identifies betas as the time series regression coefficients whose least squares estimators are used in the traditional regression approach of Fama and MacBeth (1973).

The final fourth set of conditions sets the conditional expectations of cross-sectional pricing errors to zero, i.e.

$$\mathbb{E}_{t-1}\left[e_t\right] = \mathbb{E}_{t-1}\left[r_t - \beta\lambda - \beta u_t\right] = 0. \tag{11}$$

Note that conditions (8) and (11) identify the risk premium given by $\mathbb{E}_{t-1}(r_t) = \beta\lambda$ which is of major interest in the following. We stack the $M = K(N + K_z + 1) + N$ conditions (8) to (11) as conditional moment function given by

$$g_t(x_t;\theta_0) = \begin{pmatrix} u_t \\ \text{vec}\left(u_t z_t^\top\right) \\ \text{vec}\left(e_t u_t^\top\right) \\ e_t \end{pmatrix} \tag{12}$$

with observation vector $x_t = (r_t, f_t, z_{t-1})$ and an $K(N + K_z + 1) + K$-dimensional parameter vector $\theta_0^\top = (\phi_0^\top, \phi^\top, vec(\beta)^\top, \lambda^\top)$. Note that in a typical asset pricing application, the number of test assets, N, exceeds the number of factors, K. Therefore, the model is generally over-identified because the number of moment conditions exceeds the number of parameters. In order to just identify the parameters and give particular weight to the pricing model condition (11), we weight the moment conditions with the $K \times N$ matrix $\beta^\top$. This weighting also identifies λ with its cross-sectional OLS estimate.

The conditional GMM criterion function of the asset pricing model to be minimized for estimation can then be written as

$$\mathbb{E}_{t-1}\left[g_t(x_t;\theta_0)\right]^\top \Omega \mathbb{E}_{t-1}\left[g_t(x_t;\theta_0)\right] \tag{13}$$

with weighting matrix

$$\Omega = \begin{pmatrix} I_{M-N} & 0_{(M-N)\times N} \\ 0_{N\times(M-N)} & \beta\beta^\top \end{pmatrix}. \tag{14}$$

2.3 Time-Varying Risk Premia

In the following, we want to uncover dynamic risk premia by extending the static baseline model of Section 2.1 by an observation-driven updating scheme for risk premium parameters given by

$$\vartheta_t = \omega + As_{t-1} + B\vartheta_{t-1}. \tag{15}$$

where ϑ_t is a vector containing parameters from θ_0 which suppose to vary over time. Typically, we will choose either $\vartheta_t = \lambda_t$ or $\vartheta_t = \left(\lambda_t^\top, vec(\beta_t)^\top\right)^\top$ in order to introduce dynamic risk premia.

A crucial modeling decision that must be made to use the updating scheme in (15) is the specification of s_t. This quantity is intended to provide information about the direction in which the parameters should be updated, taking into account the information at time t, represented by the most recent observations r_t and f_t. The score-driven model class of Creal et al. (2013) and Harvey (2013) employs such a scheme, using the gradient of log observation density as an innovation sequence s_t for the dynamic adjustment. In particular, Umlandt (2023) studies a score-driven model using an asset pricing framework closely related to the one in Section 2.1, which additionally needs to specify the distribution of the innovation terms u_t and e_t.

Here we want to avoid distributional assumptions about the error terms and let the dynamics be guided only by the moment conditions discussed in Section 2.2. We therefore follow the approach of Creal et al. (2024) and choose the innovation sequence s_t as the influence function of x_t on the estimator of ϑ_t from the conditional moment condition (12). Let Δ_{x_t} be the Dirac measure that puts unit mass on the actual observation x_t. Given $\epsilon \in [0,1]$, define the contaminated measure $F_x^\epsilon = (1-\epsilon)F_x + \epsilon\Delta_{x_t}$ that overweights the current observation relative to the overall observational measure F_x. GMM estimates of the time-varying parameters can then be derived based on the overall measure as $\vartheta_t(F_x)$ or based on the contaminated measure $\vartheta_t(F_x^\varepsilon)$. The influence function is then defined the limit of the (functional) difference quotient of the two estimators as ϵ tends to 0, i.e.

$$s_t = \left.\frac{d\vartheta_t(F_x)}{d\epsilon}\right|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{\vartheta_t\left(F_x^\epsilon\right) - \vartheta_t\left(F_x\right)}{\epsilon}. \tag{16}$$

Intuitively, the influence function measures the dependence of (static) parameter estimators on new observations. This concept is widely used in robust statistics to study the impact of data outliers on estimates. [1] But this also means that the influence function provides a signal of where one can get the steepest increase in the conditional GMM criterion (12) in response to the new observation x_t. Thus, instead of focusing on outliers, we use the information from the influence function to guide the updating of the time-varying risk premium parameter.

Let $\tilde{\theta}_0$ include the static parameters from θ_0 which are not covered in ϑ_t and $g_t(x_t;\vartheta_t,\tilde{\theta}_0)$ be the conditional moment criterion in (12) but including time-varying parameters ϑ_t. Creal et al. (2024) show that choosing s_t as the influence function in a restricted observation-driven updating scheme like (15) delivers a local expected improvement of the conditional criterion

[1] See Hampel et al. (2011) for a comprehensive treatment of the influence function and its use in robust statistics.

function given by

$$\mathbb{E}_{t-1}\left[g_t(x_t;\vartheta_t,\tilde{\theta}_0)\right]^\top \Omega_{t-1}\mathbb{E}_{t-1}\left[g_t(x_t;\vartheta_t,\tilde{\theta}_0)\right] \tag{17}$$

where Ω_{t-1} is a weighting matrix known at time $t-1$. We choose the dynamic weighting matrix to be either the one given in (14) or, if betas are in included in ϑ as

$$\Omega_{t-1} = \begin{pmatrix} I_{M-N} & 0_{(M-N)\times N} \\ 0_{N\times(M-N)} & \beta_t\beta_t^\top \end{pmatrix}. \tag{18}$$

Note that in the latter specification, Ω_{t-1} is still known at time $t-1$, since $\beta_t = Cov_{t-1}(r_t, u_t)\Sigma_u^{-1}$ is known at time $t-1$.

We say a common series $x_t^\top = \left(r_t^\top, f_t^\top, z_{t-1}^\top\right)$ of returns r_t, (cross-sectional) factors f_t, and time-series predictors z_{t-1} follows a **M**oment-Based **D**ynamic **A**sset **P**ricing **M**odel (MDAPM) if it fulfills the moment conditions (8) to (11) and includes an updating scheme given by (15) and (16). The following proposition presents the updating schemes for the cases in which either only risk prices or both risk prices and exposures vary over time.

Proposition 1. Let x_t follow an MDAPM(p,q).

(a) The influence function for $\vartheta_t = \lambda_t$ is given by

$$s_t^\lambda = \left(\beta^\top\beta\right)^{-1}\beta^\top r_t - \lambda_t - u_t \tag{19}$$

(b) The influence function for $\vartheta_t = \left(\lambda_t^\top, vec(\beta_t)^\top\right)^\top$ is given by

$$\begin{pmatrix} s_t^\lambda \\ s_t^\beta \end{pmatrix} = \begin{pmatrix} \left(\beta_t^\top\beta_t\right)^{-1}\beta_t^\top e_t\left(1 - u_t^\top\Sigma_u^{-1}\lambda_t\right) \\ \text{vec}\left(e_t u_t^\top \Sigma_u^{-1}\right) \end{pmatrix} \tag{20}$$

The risk price updating scheme for the case with constant betas in (Proposition 1.(a)) resembles the regression error from a (cross-sectional) regression of r_t on the risk exposures β which in the given model should provide an estimate of $\lambda_t + u_t$ because of the asset pricing restriction (11). This cross-sectional regression is also employed in the second stage of the famous two-pass approach of Fama and MacBeth (1973) who compute period-wise lambdas from those regressions and average them to achieve an estimate of the (static) risk price. Instead of averaging the regression parameters, the MDAPM quite intuitively updates time-varying risk prices based on current pricing errors. A very similar updating scheme can also be found in the SDAPM of Umlandt (2023) in which the regression follows a generalized least square fashion with $s_t^\lambda = \left(\beta^\top \Sigma_e^{-1} \beta\right)^{-1} \beta^\top \Sigma_e^{-1} e_t$ when assuming Gaussian innovations. A major obstacle to the implementation of the Gaussian SDAPM is the appearance of the typically large covariance matrix Σ_e, which renders the procedure computationally demanding. Thus, the moment-based updating scheme is on the one hand much more convenient to implement, but on the other hand may lack efficiency when heteroskedasticity and correlation in idiosyncratic errors is a prevalent feature of the data. One way to utilize this potential correlation structure is to set the lower left component of the weighting matrix in (14) to $\Sigma_e^{-1}\beta \left(\Sigma_e^{-1}\beta\right)^\top$. Weighting the risk exposures with the error covariance matrix within the moment-based framework results in the SDAPM updating scheme above.

Allowing both risk prices and exposures to vary over time leads to the update innovations shown in part (b) of Proposition 1. The risk exposure innovation $s_t^\beta = \mathrm{vec}\left(e_t u_t^\top \Sigma_u^{-1}\right)$ reflects that β can be understood as coefficients from regressing returns on factor innovations u_t while e_t is the corresponding error term. Therefore, e_t and u_t should be uncorrelated, which would mean that s_t is zero on average. If $e_t u_t^\top > 0$, we have $s_t^\beta > 0$, which leads to an increase in β (assuming the corresponding coefficient is positive). A positive score signals that there is some correlation between e_t and u_t, which means that e_t may possesses some explanatory power for returns r_t that could also be associated with u_t. Since in the any (cross-sectionally) predictable variation in returns in a factor model should be due to the factor innovations, the MDAPM increases the beta to reduce the local correlation between the different innovations and to explain the additionally found predictable variation with a higher risk factor exposure.

The risk price updating scheme in the case of time-varying risk exposures is very similar to that with constant risk exposures. However, instead of updating risk prices solely due to projection errors $\left(\beta_t^\top \beta_t\right)^{-1} \beta_t^\top e_t$, these are scaled with $\left(1 - u_t^\top \Sigma_u^{-1} \lambda_t\right)$. If $u_t = 0$, the risk price updating behaves as in the case with time-constant exposures. However, if u_t and e_t are positive, the betas are increased to remove any possible correlation. In order not to increase the risk premium $\beta_t \lambda_t$ more than proportionally, the risk price movement is damped with the factor $\left(1 - u_t^\top \Sigma_u^{-1} \lambda_t\right)$.

2.4 Higher Order Moment Conditions

The set of conditional moment restrictions used in Sections 2.2 and 2.3 to derive the moment-based risk premium filter mainly represent the first-order moment conditions. Since the interpretation of factor volatility and idiosyncratic volatility is often connected towards risk, one could suspect that second order moment conditions should be informative for the risk premium updating. We therefore consider an extension of the MDAPM with time-varying volatilities that fits an extended set of conditional moment restrictions that is represented by the following conditional moment function:

$$g_t(x_t; \theta_0) = \begin{pmatrix} u_t \\ \operatorname{vec}\left(u_t z_t^\top\right) \\ \operatorname{vec}\left(e_t u_t^\top\right) \\ e_t \\ \operatorname{vech}\left(u_t u_t^\top - \Sigma_u\right) \\ \operatorname{vech}\left(e_t e_t^\top - \Sigma_e\right) \end{pmatrix} \tag{21}$$

The following proposition presents the influence functions of the MDAPM based on the extended conditional moment conditions.

Proposition 2. Let x_t follow an MDAPM(p,q) based on the conditional moment function in (21). The influence function for $\vartheta_t = \left(\lambda_t^\top, vec(\beta_t)^\top, vech(\Sigma_{u,t})^\top, vech(\Sigma_{e,t})^\top\right)^\top$ is then given by

$$\begin{pmatrix} s_t^\lambda \\ s_t^\beta \\ s_t^{\Sigma_u} \\ s_t^{\Sigma_e} \end{pmatrix} = \begin{pmatrix} \left(\beta_t^\top \beta_t\right)^{-1} \beta_t^\top e_t \left(1 - u_t^\top \Sigma_{u,t}^{-1} \lambda_t\right) \\ \text{vec}\left(e_t u_t^\top \Sigma_{u,t}^{-1}\right) \\ \text{vech}\left(u_t u_t^\top - \Sigma_{u,t}\right) \\ \text{vech}\left(e_t e_t^\top - \Sigma_{e,t}\right) \end{pmatrix} \tag{22}$$

The general shape of MDAPM updating schemes for risk prices and exposures are not affected by additionally considering second order moment conditions. In particular, time-variation in idiosyncratic volatility, represented by $\Sigma_{e,t}$ does not affect risk premium dynamics. Thus, the derived MDAPM updating schemes are robust with respect to time- varying idiosyncratic volatility. However, factor volatility does impact risk premium dynamics as the time-varying factor covbariance matrix $\Sigma_{u,t-1}$ enters the forcing innovations s_t^λ and s_t^β. The finding that $\Sigma_{u,t}$ does not enter the risk price update in the case of time-constant betas supports the view that volatility-based factor timing strategies (Barroso and Santa-Clara, 2015; Moreira and Muir, 2017) generate returns by predicting changes in factor exposures rather than factor risk prices.

Proposition 2 states that the influence functions for the covariance matrices $s_t^{\Sigma_u}$ and $s_t^{\Sigma_e}$ are given by the difference of squared errors and the current value of the corresponding covariance matrix. Moreover, this gives the corresponding updating schemes to follow VECH processes as in Bollerslev et al. (1988). This class of models is known for its large parameterization and the challenges it poses for estimation. That is particular troublesome in the MDAPM as the number of assets N is typically large. For practical reasons, it is therefore advisable to keep the idiosyncratic covariance matrix constant, as it does not affect the dynamics of risk premia. Since the number of factors K is typically much smaller than N, a VECH specification of factor innovations is likely to be feasible. Moreover, one could also consider restricted forms of VECH, such as the diagonal VECH or the BEKK specification of Engle and Kroner (1995).

2.5 Estimation and Inference

Since the MDAPM is based on moment conditions, we follow the approach of estimating the static parameters with GMM. In addition to the model parameters in $\tilde{\theta}_0$, we need to estimate the parameters of the updating scheme collected in $(\omega^\top, \text{vec}(A)^\top, \text{vec}(B)^\top)$. Let us denote the vector containing all the remaining static parameters by θ. Since we introduced more static parameters with the recursive updating scheme than we removed by making them time-varying, and the static baseline model we started with was just identified, we now have to deal with the resulting underidentification. Therefore, we need to add additional constraints to the initial moment conditions. As suggested by Creal et al. (2024) in general[2], we define a vector of instruments $z_t = (1, s_{t-1}^\top)^\top \otimes I_{N(K+K_z+2)}$ and follow the general approach of Hansen (1982) to minimize the GMM criterion given by

$$\min_{\theta\in\Theta} \qquad \tilde{g}_T^\top \tilde{\Omega}_T \tilde{g}_T \tag{23}$$

with

$$\tilde{g}_T = \frac{1}{T}\sum_{t=1}^{T} z_t g_t(x_t; \vartheta_t, \tilde{\theta}_0) = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} 1 \\ s_{t-1} \end{pmatrix} \otimes g_t(x_t; \vartheta_t, \tilde{\theta}_0) \tag{24}$$

where $\tilde{\Omega}_T$ is a positive definite matrix weighting the moment conditions. As usual, a two-step approach can be used, where the minimization problem (23) is first solved with $\tilde{\Omega}_T = 1$. The results of the first step can be used to compute the long run variance of $g_t(x_t; \vartheta_t, \tilde{\theta}_0)$. The inverse of the long-run variance can be used as a weighting matrix in a second minimization of (23) to obtain more efficient estimates.

[2]More specifically, Creal et al. (2024) suggests using the lagged time-varying parameter as an additional instrument. We refrain from doing so because we already have a sufficient number of moment conditions by using the lagged influence functions.

Using the lagged influences s_{t-1} as instruments reduces the first-order (cross-)autocorrelation of moments which are particularly important for inferring parameter updates. For example, if betas are constant the additional moments conditions used would be $s^{\lambda}_{t-1} \otimes g_t(x_t; \vartheta_t, \tilde{\theta}_0) = (\beta^\top \beta)\beta^\top e_{t-1} \otimes g_t(x_t; \vartheta_t, \tilde{\theta}_0)$. Thus, the GMM estimator of the constant exposure MDAPM would adjust parameters to minimize the cross-correlation between lagged idiosyncratic innovations and contemporaneous moment conditions that include idiosyncratic and factor innovations as well as their products.

Creal et al. (2024) establish an asymptotic distribution theory for the general moment-based filtering framework. They show that under some high-level assumptions, it holds

$$\sqrt{T}\left(\hat{\theta} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, (G^\top \tilde{\Omega} G)^{-1} G^\top \tilde{\Omega} S \tilde{\Omega}^\top G (G^\top \tilde{\Omega} G)^{-1}\right), \tag{25}$$

where G is the limit of the gradient of (23) and the asymptotic covariance matrix $S = \sum_{j=-\infty}^{\infty} \mathbb{E}\left(g_t g_{t-j}^\top\right)$ of the moment conditions. The required high-level assumptions include that the filter of the time-varying parameter vector ϑ_t converges to a unique stationary and ergodic solution. For the case that betas are constant, i.e. $\vartheta_t = \lambda_t$, it can be shown straightforwardly that the stationarity assumption is fulfilled if $\|B - A\| < 1$ and $\|(\beta^\top \beta)^{-1}\| < \infty$.[3] Whereas the first inequality rules out explosive behavior of risk premia, the second one is met in absence of weak factors. Similar parameter restrictions are much more cumbersome to derive in the case with time-varying betas as risk premia are then a product of distinct time-varying parameters.

We use the result in (25) in the following application to compute standard errors for the parameter estimates, where the asymptotic covariance matrix is estimated with the heteroscedasticity and autocorrelation consistent estimator from Newey and West (1987) with Bartlett kernel as in Andrews (1991).

[3] $\|A\|$ denotes the spectral norm of a matrix A.

3 Simulation Study

In this section, the small sample performance of the MDAPM is evaluated with a Monte Carlo study on risk premium filtering.

3.1 Data-Generating Process

Assume there is one (cross-sectional) risk factor f_t whose factor mean can be predicted by the univariate series z_t with

$$f_t = z_{t-1} + u_t \tag{26}$$

where u_t follows a GARCH(1,1)-process with an unconditional variance of 18. The risk factor calibration is chosen to represent the features of the CRSP market return and the DGP can therefore be interpreted as a CAPM with dynamic coefficients. The factor mean predictor z_t is simulated from the AR(1) model given by

$$z_t = 0.5 + 0.98(z_{t-1} - 0.5) + \varepsilon_{z,t}, \qquad \varepsilon_{z,t} \sim \mathcal{N}\left(0, 0.137^2\right) \tag{27}$$

with its initial value z_1 drawn from the corresponding unconditional distribution. The process in (27) is calibrated in order to represent the moments and persistence of equity return predictors used in the following empirical application. Especially the highly persistent autoregressive parameter of 0.98 is often observed for factors considered as market return predictors (Campbell and Yogo, 2006).

The simulated returns r_t are derived from the beta representation according to (7) with N assets in the panel and T observations over time. Idiosyncratic innovations $e_{i,t}$ are drawn from a a Student-t-distribution with $\nu = 6$ degrees of freedom and a variance of 4. The betas are simulated from a slow-moving processes given by

$$\beta_{i,t} = \overline{\beta}_i + 0.95(\beta_{i,t-1} - \overline{\beta}_i) + 0.05\varepsilon_{\beta_i,t}, \qquad \varepsilon_{\beta_i,t} \sim \mathcal{N}\left(0, 0.148^2\right). \tag{28}$$

where the N unconditional exposures $\overline{\beta}_i$ form an equidistant grid of the interval $[0.6, 1.5]$. Parameters in the process (28) are based on the results of the following empirical application. The unconditional beta interval is approximately the range observed for exposures when industry portfolio returns[4] are regressed on the market risk factor and a constant.

Since the DGP contains only one pricing factor, we need to simulate only one risk price λ_t. We consider four alternative settings for the dynamics of the risk price:

$$\text{Constant:} \qquad \lambda_t = 0.5 \tag{29}$$

$$\text{Cycle:} \qquad \lambda_t = 0.5 + \sin(2\pi t/T) \tag{30}$$

$$\text{Breaks:} \qquad \lambda_t = \begin{cases} 0.5, & \text{if } t \in [0, T/3] \\ -0.5, & \text{if } t \in (T/3, 2T/3] \\ 1.5, & \text{if } t \in (2T/3, T] \end{cases} \tag{31}$$

$$\text{AR:} \qquad \lambda_t = 0.5 + 0.98(\lambda_{t-1} - 0.5) + \varepsilon_{\lambda,t}, \qquad \varepsilon_{\lambda,t} \sim \mathcal{N}\left(0, 0.25^2\right) \tag{32}$$

The first constant risk price DGP serves as a benchmark to evaluate the performance of the dynamic MDAPM, since the actual process is static. Next, the cycle DGP reflects the idea that risk prices move with the business cycle. The third DGP with breaks is intended to mimic a situation in which unexpected news immediately changes investors' perception of risk and, therefore, the price of risk demanded. Finally, the fourth process represents the situation where risk premia follow a highly persistent process, such as the betas in the employed DGP do.

[4] Although industry portfolios are not used in the empirical application, we use their exposure range for calibration in order to have a higher degree of dispersion in the betas to explore in simulations.

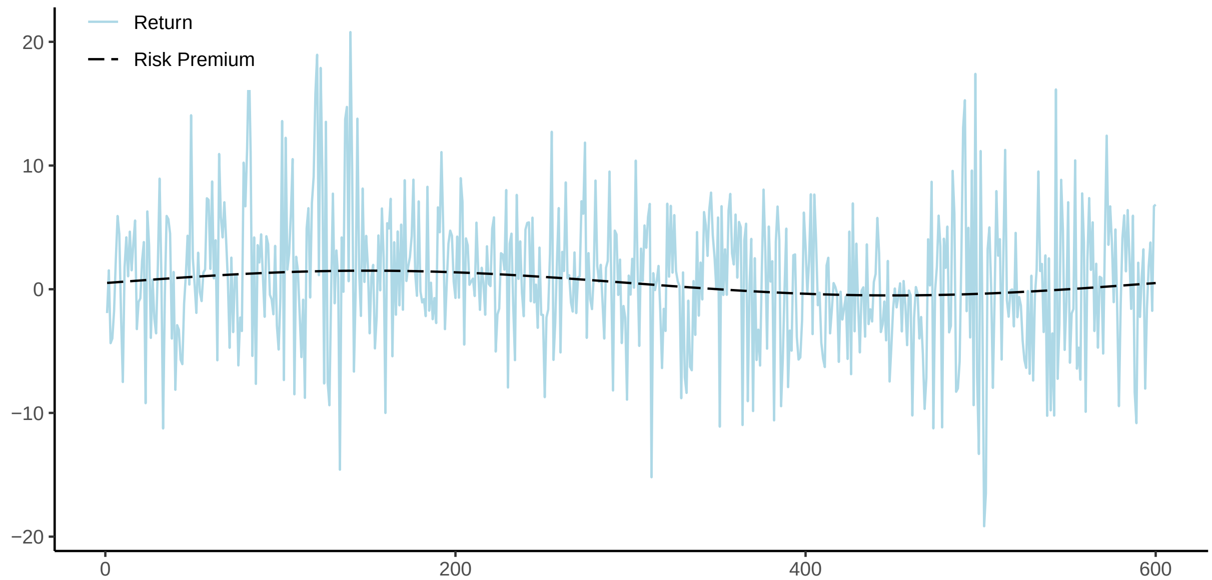

Figure 1: **Simulated Returns.** This figure shows simulated excess returns from an asset with unit (unconditional) risk factor exposure together with the cyclical risk premium λ_{t-1} from one draw of a panel with $N = 25$ assets and $T = 600$ time observations.

Figure 1 shows an example draw of the simulated excess return of an asset with unit (unconditional) risk factor exposure together with the cyclical risk premium λ_{t-1} from a panel of size $N = 25$ and $T = 600$. The cyclical variation in the conditional return expectation can be considered small compared to the variation in the realized return. This is consistent with the perception that stock returns have little predictability and poses a challenge to our method that attempts to filter out these almost diminishing dynamics. Also consistent with the stylized statistical facts of financial returns, the series shows volatility clustering, which in the DGP used comes from the GARCH residual of the pricing factor process in (26).

3.2 Simulation Results

In the following, we discuss the results of fitting a MDAPM with time-varying lambdas and betas to $S = 1000$ Monte Carlo replications of each of the four DGPs mentioned above.

3.2.1 Predicted Risk Prices

Figure 2 shows, for a panel of size $N = 25$ and $T = 600$, the average risk price predicted by the MDAPM (solid line) together with the true risk price (dashed line). The shaded areas represent the 90 percent interquantile bands. We see that the MDAPM is able to adequately track the true risk price process in all three cases, on average with an interquantile band range of about 0.9. Given the rather low signal-to-noise ratio and the high variance of innovations, the filter uncertainty reflected by the bands can be considered rather low. In the case of a constant true risk price, the MDAPM is able to filter almost perfectly on average. In the cyclical case, the MDAPM tracks the true risk price with a short lag, as is typical for observation-driven models that process new information with a time lag. Similarly, the moment-based filter can react to breaks in the third DGP, but with a short delay. In the final AR case, the MDAPM faces the most difficulties. Although the filter correctly anticipates the direction of the true process and tracks it fairly well, it barely covers short-term peaks.

3.3 Pricing and Prediction Error Comparison

We further investigate the performance of the MDAPM with respect to pricing and prediction errors which are summarized in Table 1. The first metric for evaluating the pricing performance is the root mean squared pricing error (RMSE), which is computed as

$$RMSE_i = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \hat{e}_{i,t}^2}. \tag{33}$$

For comparability, we compute ΔRMSE as the difference between the RMSE of Fama and MacBeth (1973) regressions (hereafter FMB) assuming constant risk premia and the RMSE produced by the MDAPM. Thus, a positive ΔRMSE indicates that the MDAPM outperforms the FMB benchmark in terms of pricing errors produced. Panel (a) in Table 1 shows the ΔRMSE averaged over portfolios and Monte Carlo replications for panels with different cross section sizes $N = 10, 25, 100$ and time series lengths $T = 300, 600, 1200$. We clearly see

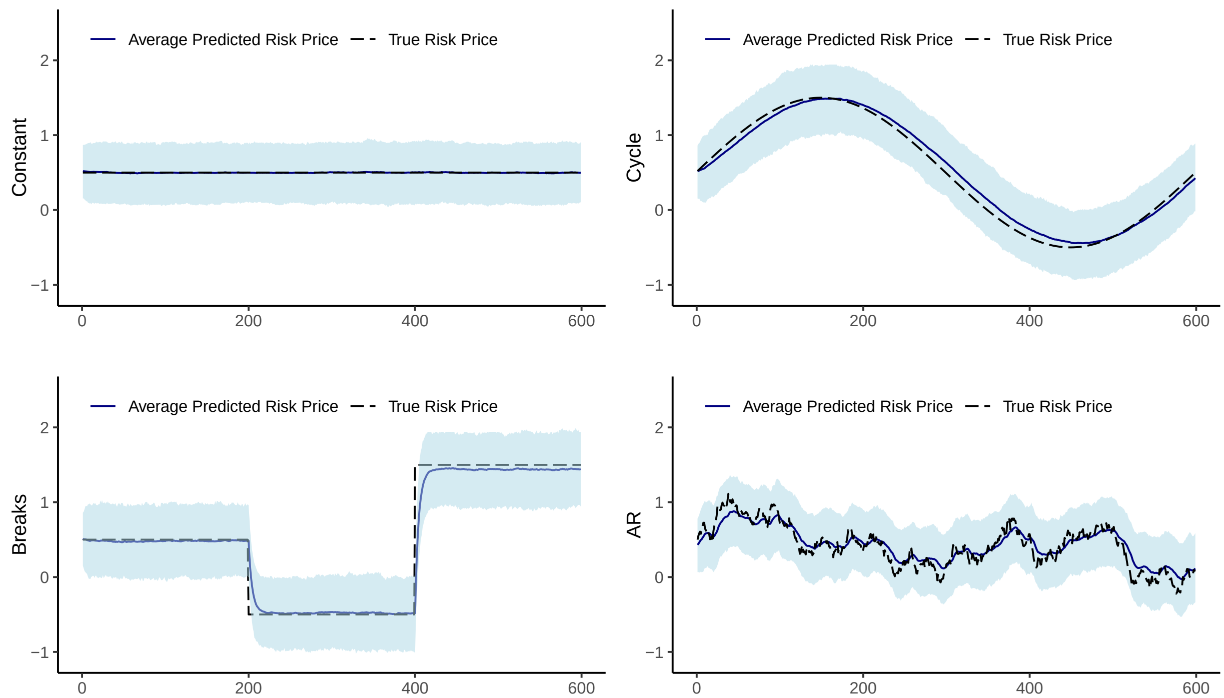

Figure 2: **Risk Price Predictions.** This figure shows, for a panel of size $N = 25$ and $T = 600$, the average risk price predicted by the MDAPM (solid line) together with the true risk price (dashed line). The shaded areas represent the 90 percent bands.

that the MDAPM outperforms the static FMB benchmark in every panel and DGP. This is particularly true for the constant risk price DGP, where the MDAPM benefits from the ability to account for time-varying betas and factor means compared to the FMB approach. As expected from Figure 2, the MDAPM performs worst for the fourth DGP with persistent autoregressive risk prices, although it still outperforms the benchmark. The latter benefits in this case especially from the low variability of risk prices. When comparing the performance of different panel sizes, we observe that the MDAPM performance improves monotonically with longer panels, i.e. higher T, but not necessarily with larger cross-sections, i.e. higher N. An explanation is that as the number of assets in the panel increases, additional parameters need to be estimated, namely the additional betas that are associated with the new assets. In contrast, increasing the length of the time series does not introduce additional parameters to the model and enables more accurate estimation due to improved data coverage.

Table 1: Pricing and Prediction Error Comparison

The table shows difference root mean squared pricing errors ($\Delta RMSE$) and root mean squared prediction errors ($\Delta RMSPE$) of a MADPM(1,1) with Fama and MacBeth (1973) regressions as benchmark averaged across assets and Monte Carlo replications. The nine simulated panels have different numbers of assets N, time observations T and are replicated 1000 times each.

	N=10			N=25			N=100		
	T= 300	600	1200	T=300	600	1200	T=300	600	1200
(a) Average ΔRMSE									
Const	0.0796	0.0948	0.1055	0.0925	0.1043	0.1096	0.0977	0.1035	0.1067
Cycle	0.1564	0.1913	0.2144	0.1636	0.1981	0.2072	0.1547	0.1562	0.1788
Breaks	0.2093	0.2322	0.2468	0.1988	0.2341	0.2525	0.1981	0.2016	0.2049
AR	0.0639	0.0939	0.1080	0.0808	0.1041	0.1117	0.0762	0.0934	0.1077
(b) Average ΔRMSPE									
Const	0.0031	0.0010	0.0004	0.0038	0.0014	0.0008	0.0044	0.0018	0.0010
Cycle	0.0499	0.0487	0.0488	0.0528	0.0493	0.0497	0.0554	0.0512	0.0502
Breaks	0.0634	0.0635	0.0629	0.0657	0.0627	0.0642	0.0662	0.0621	0.0637
AR	0.0084	0.0064	0.0046	0.0107	0.0077	0.0067	0.0111	0.0090	0.0071

The second metric we evaluate is the the root mean squared prediction error (RMSPE):

$$RMSPE_i = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(r_t - \hat{\beta}_{i,t}\hat{\lambda}_t\right)^2}. \tag{34}$$

This measure captures how well the model fits that the conditional expectation of r_{t+1} is given by $\beta_t\lambda_t$. Again, we construct a difference measure ΔRMSPE with a FMB baseline model to compare results from different panels. The averaged ΔRMSEs are shown in Panel (b) of Table 1. We observe that the MDAPM is also superior to the static FMB benchmark in every panel and DGP with respect to prediction errors. In contrast to the $\Delta RMSE$ results, we see that the performance of the MDAPM with respect to the static benchmark improves with increasing N. Hence, it seems that estimating the risk premium, whose dimension is unchanged, benefits from a richer cross-section, but other parameters are estimated with greater uncertainty, thus increasing idiosyncratic pricing errors. The rather counterintuitive result that the ΔRMSPEs decrease with T is technical. The beta processes in the DGP exhibit fairly high persistence, leading to large and long-lasting deviations from the unconditional mean. These deviations, which become more pronounced in longer samples, are generally more difficult to track than movements closer to the unconditional mean.

In summary, the MDAPM is capable of tracking the variation of risk premia with various dynamics in a setting with a realistically low signal-to-noise ratio. Its pricing performance, in particular, benefits from longer time series. Additionally, the risk premium prediction improves with larger cross sectional dimensions.

4 Empirical Application

The following empirical application examines the dynamics of risk premia that can be derived from the asset pricing moments in the 3-factor model of Fama and French (1993).

4.1 Data

The test assets are 25 stock portfolios sorted by size and book-to-market equity (value). The monthly series are obtained from Kenneth French's online library and cover the period from January 1964 to June 2023. Thus, we work with a return panel with dimensions $N = 25$ and $T = 691$. Three risk factors are considered to price this cross-section of test assets $f_t = (MKT_t,\ SMB_t,\ HML_t)^\top$ where MKT is the excess return on the value-weighted equity market portfolio, SMB and HML are the small minus big and high minus low portfolio returns from Fama and French (1993).

As forecast instruments for the conditional factor mean we use the three-dimensional vector $z_t = (TSY10_t,\ TERM_t,\ DY_t)^\top$ where TSY10 is the 10-year treasury yield and TERM is the term spread, calculated as the difference between the yields of the 10-year treasury note and the three-month treasury bill. Both series are obtained from the H.15 statistical release of the Board of Governors of the Federal Reserve System. The third forecasting factor DY is the dividend yield of the S&P 500 index. Evidence on equity return predictability from these factors can be found in Keim and Stambaugh (1986), Campbell (1987), Fama and French (1989), and Campbell and Thompson (2008) for long-run treasury yields and Campbell and Shiller (1988), Fama and French (1989), Campbell and Thompson (2008), and Cochrane (2008) for the term structure and dividend yields.

4.2 Empirical Model Specifications

The main specification to be considered is an MDAPM with time-constant betas. The parameter matrices A and B in the updating scheme are assumed to be diagonal in order to achieve a updating equation given by

$$\lambda_t^j = \overline{\lambda}^j + a_j^\lambda s_{j,t-1}^\lambda + b_j^\lambda \left(\lambda_{t-1}^j - \overline{\lambda}^j \right) \tag{35}$$

for every cross-sectional pricing factor $j = MKT, SMB, HML$. Note that the updating equations are parameterized along their long-run values $\overline{\lambda}^j$ which can be interpreted as risk prices of the corresponding factor j. The diagonalization mutes the impact of the influence function on the parameter updating of the other factor. However, excluding those effects in the present application does not crucially impair the model performance but rather yields a much more parsimonious model.

We also consider an MDAPM specification with time-varying betas. We further assume that the diagonal parameters for the beta update of the exposure of asset i to factor j are the same for each asset i. This assumption allows estimation of the model with beta dynamics for larger panels and follows the assumption that exposures to the same factor follow similar dynamics. The resulting updating equations, parameterized along the long-run values $\overline{\beta}$, are given by

$$vec\left(\beta_{i,t}^{j}\right) = vec\left(\overline{\beta}_{i}^{j}\right) + a_{j}^{\beta} s_{i,j,t-1}^{\beta} + b_{j}^{\beta} vec\left(\beta_{i,t-1}^{j} - \overline{\beta}_{i}^{j}\right) \tag{36}$$

with parameters $\overline{\beta}^{j}$, a_{j}^{β}, and b_{j}^{β}, where $i = 1, \ldots, 10$ and $j = MKT, SMB, HML$.

Two established benchmark specifications are considered. The first benchmark is the unconditional risk price specification underlying classical Fama and MacBeth (1973) regressions. In line with Adrian et al. (2015), estimated innovations $\hat{u}_t$ from a VAR(1) model including the abovementioned factors are provided as pricing factors in order to account for the significant autocorrelation within the pricing factors. The second benchmark is a DAPM that explains risk price variations with the forecasting factors described above. This yields a regression equation given by

$$\lambda_{t}^{j} = \lambda_{0} + \Lambda_{1}^{j,TSY10} TSY10_{t-1} + \Lambda_{1}^{j,TERM} \Delta TERM_{t-1} + \Lambda_{1}^{j,\Delta DY} \Delta DY_{t-1} \tag{37}$$

for each of the two cross-sectional risk factors $j = MKT, SMB, HML$. Estimation and inference for the DAPM is done as described in Adrian et al. (2015), and I refer to them for more details.

4.3 Empirical Results

In the following, we present the empirical results of the application to the Fama-French 3-factor model regarding parameter estimates, filtered risk premia, and pricing errors.

4.3.1 Parameter Estimates

Parameter estimates for the risk premium updating schemes are shown in Table 2. The first row shows the estimated unconditional risk prices $\overline{\lambda}$ of the two MDAPM specifications and the two benchmarks. We see that the average risk prices in the MDAPM differ moderately from those in the static unconditional specification. The standard errors are shown in the

Table 2: Risk Premium Parameter Estimates

This table shows estimates of risk premium parameter estimates for the market factor (MKT), the small-minus-big factor (SMB), and the high-minus-low (HML) factor. The first three columns show results from a MDAPM with constant betas. Following three columns show results from a MDAPM with time-varying betas is a specification with unique learning and persistence rates per factor for the beta updating. Columns 7 to 9 provide estimates from a dynamic asset pricing model (DAPM) in line with Adrian et al. (2015). Forecasting factors are the 10-year treasury yield (TSY10), the term spread (TERM), and the dividend yield of the S&P 500 index (DY). The last three columns provide regression results of a constant (unconditional) risk price specification. MDAPM standard errors shown in parentheses are derived as described in Section 2.5. Errors for the DAPM estimates are adjusted for cross-asset correlation in the residuals and for estimation error of the time-series betas. GMM standard errors for the unconditional specification. Test assets are 25 value-weighted equity portfolios sorted on size and value with monthly returns denoted in percentages. The sample period is 1964:01 - 2023:06.

	MDAPM			MDAPM			DAPM			Unconditional		
	MKT	SMB	HML	MKT	SMB	HML	MKT	SMB	HML	MKT	SMB	HML
$\overline{\lambda}$	0.578	0.210	0.381	0.478	0.282	0.369	0.526	0.232	0.337	0.528	0.233	0.338
	(0.549)	(0.240)	(0.216)	(0.745)	(0.335)	(0.349)	(0.150)	(0.145)	(0.178)	(0.171)	(0.118)	(0.116)
a^{λ}	0.362	0.411	0.091	0.297	0.244	0.084						
	(0.167)	(0.191)	(0.132)	(0.065)	(0.104)	(0.114)						
b^{λ}	0.960	0.687	0.693	0.904	0.545	0.714						
	(0.110)	(0.345)	(1.018)	(0.149)	(0.603)	(0.836)						
a^{β}				0.078	0.013	0.034						
				(0.015)	(0.007)	(0.012)						
b^{β}				0.735	0.977	0.971						
				(0.186)	(0.070)	(0.034)						
TSY10							-0.249	-0.106	0.156			
							(0.086)	(0.061)	(0.060)			
TERM							0.204	0.096	0.057			
							(0.137)	(0.096)	(0.092)			
DY							1.689	1.005	-0.807			
							(0.638)	(0.463)	(0.452)			

rows below the parameter estimates. Note that average risk prices are generally estimated with more uncertainty in the dynamic model variants, due to the increased number of parameters that these models have to fit. Regarding the learning rates of the risk prices a^{λ}, we find that they are significantly positive for MKT and SMB in both variants with either constant or time-varying betas. This means that the premium for the MKT and SMB factors appears to vary significantly over time. For the HML risk price, we find that the learning rate is positive but insignificant. This indicates that the value premium appears to be rather stable over time.

The persistence rates b^{λ} indicate that the price of market risk moves very persitently while SMB and HML risk price series only moderately depend on their prior values. When risk exposures are time-varying, b^{λ} drops drastically for SMB and slightly for HML. This, persistent dynamics in size risk premia likely stem from movement in risk exposures rather than risk prices. The risk price parameters a^{λ} and b^{λ} do not crucially change when allowing for time-varying risk exposures. However, the long-run risk price $\overline{\lambda}$ clearly drops from 0.578 to 0.478. In general, we note that the risk price dynamics are affected by the dynamics in risk exposures and should not be analyzed in isolation.

With respect to the persistence and learning rates for the beta series (a^{β} and b^{β}), we see that MKT and HML betas appear to significantly vary over time while a^{β} is positive but not significantly different from zero for the SMB factor. The persistence rates are b^{β} are close to on for SMB and HML while it is rather moderate for MKT. This is particularly reasonable because of the sorting of the test assets. Since the portfolios are sorted by their market capitalization and book-to-market ratio, the risk exposure is expected to be quite stable with respect to the SMB and HML factor.

The estimated coefficients of DAPM are comparable to those documented in Adrian et al. (2015), but the term spread appears to be less relevant in the 25 Fama-French portfolios than it is in their combined equity and bond cross-section.

4.3.2 Filtered Risk Premia

Figure 3 shows the filtered risk prices of the MDAPM specification alongside the risk prices proposed by the DAPM and the constant FMB. For the MKT risk price in the upper panel of 3, we see that the two dynamic approaches tend to move roughly together. In general, the DAPM risk price seems to be slightly higher in periods of high risk premiums, such as

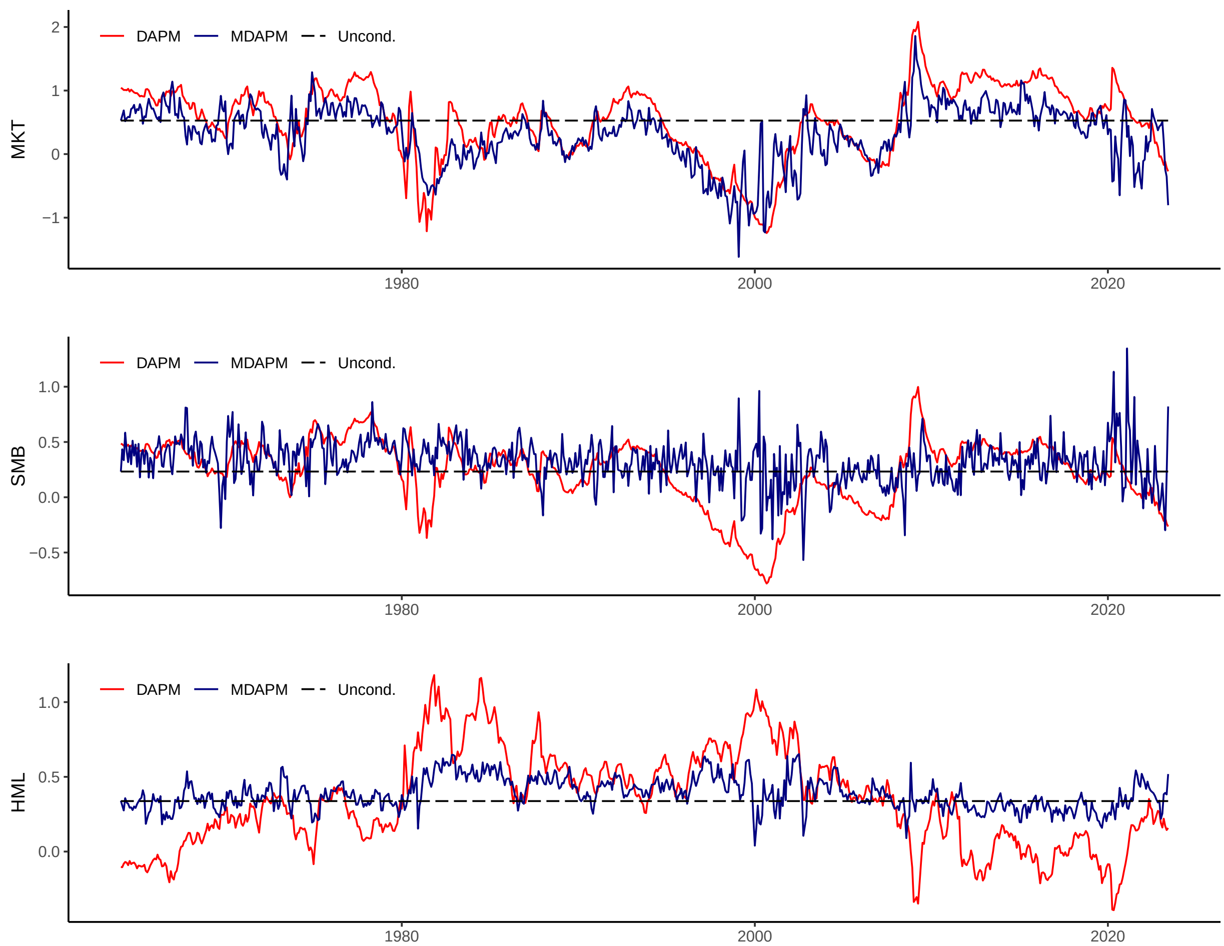

Figure 3: **Time-Varying Risk Prices.** This figure shows estimated factor risk prices for the market factor (MKT), the small-minus-big factor (SMB) and the high-minus-low (HML) factor. It reports results from MDAPM with time-varying exposures and a DAPM from Adrian et al. (2015). Test assets are 25 value-weighted equity portfolios sorted on size and value with monthly returns denoted in percentages. The sample period is 1964:01 - 2023:06.

the second half of the 1970s and the 2010s. Notably, the moment-based risk prices are more volatile. Moreover, we can also find that risk premia tend to increase in recessionary periods when markets are in a downturn. This countercyclicality has already been documented with regression-based approaches in Adrian et al. (2015), Gagliardini et al. (2016, 2020), and Chaieb et al. (2021). At the same time, we also see the pattern that market risk premia tend to fall at the beginning of crisis periods, as can best be seen around the global financial crisis. This pattern has also been documented in Jensen (2018), Gómez-Cram (2022) and Umlandt (2023). One reason could be that time-varying lambdas reflect not only the price of risk, but also the expected return. Thus, market lambdas may decline at the beginning of a recession due to lower expectations. In contrast to the results of the likelihood-based filter in Umlandt (2023), we see that although the market risk premium falls at the beginning of a recession, it rises quickly after the first drop, as can best be seen again in the global financial crisis. Thus, it appears that after an initial adjustment of expectations, adverse events cause investors to demand higher compensation for risk, in line with the previously found countercyclicality.

We see that the MDAPM and DAPM suggest very different SMB risk price trajectories from the 1980s to the recent past when we look at the SME risk prices shown in the middle panel of Figure 3. While the DAPM suggests a fairly persistent, slow-moving risk premium that follows almost the same dynamics as the market premium, the MDAPM-filtered size premium is much more volatile, especially around 2000. For the HML price shown in the lower panel of Figure 3, we find that the MDAPM implied risk prices moves only slightly. This is in line with the small and insignificant learning rate a^{λ} found in Table 2. A reasoning for the observation that the two approaches mostly agree in the case of the market risk premium is that the risk price predictors used in the DAPM are especially documented to predict the market return as can be also seen by the coefficient estimates of the DAPM in Table 2. These predictors appear to adequately span the information set for the conditional price of market risk, even though they may not be sufficient to span the information set for the other two pricing factors. Therefore, it is likely that more appropriate instruments can be found to study the dynamics of SMB and HML risk pricing in a regression-based framework.

4.3.3 Pricing and Prediction Error Comparison

We evaluate the in-sample pricing and premium prediction performance according to two measures. The first measure for evaluating the pricing performance is the root mean RMSE, as computed in (33). Columns 1 through 4 of Table 34 show the RMSE for the 25 portfolios individually as well as averaged over the entire set of test assets. From the averages, we can see that the moment-based MDAPM filter with constant exposures largely reduces the average RMSE from 1.781 to 1.687, while the regression-based DAPM benchmark's RMSE is only reduced to 1.755. Also from an asset-specific perspective, we find that the MDAPM crucially reduces the pricing error in comparison to DAPM and FMB for most portfolios. The pricing error reduction is particularly high for the small size portfolios s1v1 to s1v5. Note also that allowing risk exposures to vary over time, as seen in the results in column 2, yields additional pricing error improvements compared to the constant β MDAPM specification.

The second metric we evaluate is the the RMSPE defined in (34). Again, this measure captures how well the model fits that the conditional time $t-1$ expectation of r_t is given by $\beta_t \lambda_t$. This is because the conditional expectation of u_t should be zero. Because the variance of the factor innovation is typically relatively high compared to that of the idiosyncratic observations, the RMSPEs shown in columns 5 through 8 of Table 34 are considerably larger than the RMSEs. In addition, the difference between the conditional model specification and the unconditional model specification is much smaller. The smallest errors in predicting the risk premium, with an average of 5.661, are found for the regression-based DAPM in column 8. The average RMSPE of the constant risk exposure MDAPM is 5.669, which is slightly higher than for the DAPM benchmark. One explanation for the relatively better performance of the regression-based benchmark in terms of RMSPE, as opposed to RMSE, is that the DAPM estimates risk prices by regressing the earlier estimated product $\beta\lambda_t$ on the forecast factors z_{t-1}. Thus, the approach is particularly suited to adjusting risk prices in a way that minimizes the RMSPE. However, this also results in less accurate fitting of other moments such as those captured by the RMSE. Comparing the average RMSPEs of the dynamic models with that of the unconditional FMB, which is given by 5.705, we find that the MDAPM does not perform significantly worse than the DAPM. The introduction of time-varying betas increases the RMSPE relative to the constant beta variant. Thus, the slight improvement in the RMSE from allowing time-varying betas seems to come at the

Table 3: Root Mean Squared Pricing and Prediction Errors

The table shows root mean squared pricing errors (RMSE) and root mean squared prediciton errors (RMSPE) of different asset pricing model specifications. Pricing factors are the market factor (MKT), the small-minus-big factor (SMB) and the high-minus-low factor (HML). MDAPM refers to results from specifications in which risk exposures are either constant or time-varying. DAPM refers to a dynamic asset pricing model specification according to Adrian et al. (2015) using the 10-year treasury yield (TSY10), the term spread (TERM), and the dividend yield of the S&P 500 index (DY) as risk price predictors. Uncond. refers to errors from constant lambda specifications estimated with Fama and MacBeth (1973) regressions. Test assets are 25 value-weighted equity portfolios sorted on size and value with monthly returns denoted in percentages. sivj refers to the portfolio of stocks in the intersection of the i-th quintile portfolio sorted on size and the j-th quintile portfolio sorted on value. The sample period is 1964:01 - 2023:06.

	RMSE				RMSPE			
	MDAPM (const. β)	MDAPM (t.v. β)	DAPM	Uncond.	MDAPM (const. β)	MDAPM (t.v. β)	DAPM	Uncond.
s1v1	2.586	2.564	2.895	2.964	8.099	8.105	8.059	8.150
s1v2	1.984	1.910	2.222	2.270	7.052	7.054	7.024	7.064
s1v3	1.339	1.362	1.601	1.645	6.091	6.102	6.076	6.121
s1v4	1.348	1.350	1.607	1.645	5.862	5.874	5.853	5.877
s1v5	2.120	2.163	2.389	2.427	6.351	6.367	6.333	6.381
s2v1	1.691	1.702	1.779	1.837	7.272	7.283	7.235	7.326
s2v2	1.318	1.311	1.447	1.484	6.172	6.186	6.163	6.212
s2v3	1.444	1.332	1.489	1.513	5.584	5.601	5.573	5.625
s2v4	1.308	1.292	1.335	1.350	5.406	5.423	5.401	5.440
s2v5	1.400	1.400	1.523	1.543	6.317	6.335	6.321	6.361
s3v1	1.605	1.616	1.606	1.642	6.678	6.692	6.647	6.724
s3v2	1.522	1.493	1.540	1.559	5.636	5.656	5.630	5.677
s3v3	1.565	1.405	1.576	1.590	5.154	5.168	5.151	5.194
s3v4	1.504	1.437	1.510	1.523	5.204	5.222	5.204	5.245
s3v5	1.915	1.865	1.896	1.907	5.978	5.995	5.978	6.009
s4v1	1.568	1.560	1.621	1.646	5.992	6.012	5.978	6.023
s4v2	1.664	1.488	1.693	1.713	5.323	5.339	5.318	5.359
s4v3	1.739	1.555	1.734	1.742	5.098	5.113	5.099	5.128
s4v4	1.766	1.716	1.736	1.744	5.143	5.153	5.137	5.168
s4v5	2.080	2.077	2.096	2.102	5.932	5.947	5.937	5.961
s5v1	1.191	1.135	1.208	1.224	4.830	4.855	4.806	4.838
s5v2	1.509	1.371	1.482	1.489	4.539	4.562	4.523	4.551
s5v3	1.721	1.641	1.701	1.703	4.481	4.497	4.472	4.497
s5v4	1.699	1.573	1.648	1.651	4.912	4.913	4.904	4.911
s5v5	2.586	2.545	2.602	2.610	5.784	5.781	5.795	5.811
Avg	1.687	1.635	1.757	1.781	5.796	5.809	5.785	5.826

cost of a slightly worse risk premium prediction error.

In summary, the error comparisons suggest that the moment-based approach can perform an estimation of dynamic asset pricing models with a crucially improved fit compared to both the conditional and the unconditional benchmark. It appears that the most substantial improvement comes from the presence of time-varying risk prices. Admittedly, it may be that better performance can be achieved by using individual updating schemes for the betas instead of specifying unique learning and persistence rates. However, this is challenging to implement as the number of parameters would increase massively and may tend to introduce more overfitting, as already suggested by the relatively high prediction errors of the specification used.

4.3.4 Alternative Factor Models and Cross-Sections

The Fama-French 3-factor model (FF3), together with the 25 test assets sorted by size and book-to-market value, is a suitable test application for the MDAPM because the factors cover a large portion of the cross-sectional variation and the portfolios have a strong factor exposure. Therefore, we further investigate the potential of the MDAPM to uncover risk premia dynamics on a set of alternative factors and test assets. As alternative factor models we consider the CAPM that only includes the market factor MKT that is also included in FF3, as well as the Carhart (1997) model (FFC) that includes a momentum factor in addition to the three factors in FF3. The momentum signal is based on the prior returns in $t-12$ to $t-1$. Besides the 25 portfolio cross-section sorted by size and book-to-market (BM) ration, we consider a merged cross section of these 25 portfolios and 10 portfolios sorted by momentum, as well as cross-section of 25 portfolios double-sorted by size and momentum. The additional data have been download from Kenneth French's data library.

Table 4 shows average RMSEs and RMSPEs for the alternative factor models and test assets. Panel (a) shows the results for the three different factor models on the panel of 25 portfolios sorted by size and BM that have been investigated in the previous section. With regard to the CAPM specification we see that the dynamic approaches are not able to improve upon the unconditional benchmark. This is not surprising given the rather low cross-sectional explanatory power of the CAPM with a cross-sectional R^2 of 24.97 percent (compared to

Table 4: Error Comparisons for Alternative Factor Models and Cross-Sections

The table shows average root mean squared pricing errors (RMSE) and root mean squared prediciton errors (RMSPE) of different asset pricing model specifications and cross-sections. Factor models are the CAPM including only MKT, the FF3 model including MKT, SMB and HML, and the FFC model including the FF3 factor plus a momentum factor. MDAPM refers to results from specifications in which risk exposures are either constant or time-varying. DAPM refers to a dynamic asset pricing model specification according to Adrian et al. (2015) using the 10-year treasury yield (TSY10), the term spread (TERM), and the dividend yield of the S&P 500 index (DY) as risk price predictors. Uncond. refers to errors from constant lambda specifications estimated with Fama and MacBeth (1973) regressions. The sample period is 1964:01 - 2023:06.

	RMSE				RMSPE			
	MDAPM (const. β)	MDAPM (t.v. β)	DAPM	Uncond.	MDAPM (const. β)	MDAPM (t.v. β)	DAPM	Uncond.
(a) 25 Size×BM								
CAPM	2.982	3.184	2.977	2.959	5.826	5.830	5.788	5.828
FF3	1.687	1.635	1.757	1.781	5.796	5.809	5.785	5.826
FFC	1.697	1.687	1.757	1.790	5.804	5.814	5.784	5.826
(b) 25 Size×BM + 10 Momentum								
CAPM	2.890	3.001	2.881	2.867	5.744	5.751	5.711	5.752
FF3	1.932	1.863	1.978	1.997	5.716	5.728	5.709	5.751
FFC	1.715	1.696	1.770	1.799	5.725	5.727	5.706	5.749
(b) 25 Size×Momentum								
CAPM	3.211	3.170	3.147	3.128	6.025	6.044	6.002	6.054
FF3	2.377	2.297	2.409	2.435	6.009	6.026	6.000	6.053
FFC	1.817	1.805	1.856	1.891	6.014	6.062	5.991	6.047

a cross-sectional R^2 of 64.64 percent in FF3). The MDAPM appears to be incapable of filtering out time series dynamics that are obscured by uncaptured cross-sectional variations. Applying the FFC model to the 25 Size×BM portfolio panel yields a significantly positive risk premium for the momentum factor but no improvements of the pricing errors in the unconditional setting. The inclusion of the additional factor in FFC does not noticeably affect the performance of the dynamic specifications compared to the FF3 model. Particularly in the MDAPM specifications, the risk price of the momentum factor seems to be quite stable.

Panel (b) shows pricing errors based on a test asset panel with a total of 35 portfolios, 25 of which are bivariate sorts with respect to size and BM, and the other 10 are univariate sorts with respect to momentum. Not too surprisingly, we see that the inclusion of the momentum factor in the FFC model clearly reduces pricing errors compared to the FF3 model. The improvement of the MDAPM specifications in relation to the two benchmarks is similar to that in the size$x$BM panel. Furthermore, the risk price of momentum is also found to be fairly stable over the period, although the momentum factor exposures show significant movements. This suggests that the lack of momentum risk price movements in the cross section of panel (a) is not necessarily due to low exposure to this factor. In fact, we also find a rather small gain from including factor premium dynamics in the cross-section of panel (b), which includes portfolios explicitly constructed according to the momentum signal. Panel (b) presents the results for a panel of double-sorted portfolios based on size and momentum, which confirm the same findings.

5 Conclusions

We introduced with the MDAPM a GMM-based dynamic asset pricing framework for linear factor pricing models. Time-varying risk premia are derived from an updating scheme that seeks a steepest descent improvement of the local GMM criterion function of a parsimonious set of asset pricing moments in the corresponding time period. It turns out that such a constructed updating mechanism adjusts risk prices according to regression errors from the cross-sectional regression performed in the second stage of the FMB procedure. In the case of time-varying betas, these are updated to enforce orthogonality of factor innovations and idiosyncratic innovations.

The MDAPM is applicable to a wide range of factor asset pricing models, does not require the specification of time series predictors, and does not require the specification of residual distributions. It is therefore a fairly robust alternative to the regression- and likelihood-based approaches of Adrian et al. (2015), Gagliardini et al. (2016), and Umlandt (2023), respectively. Estimation and inference can be performed in a standard GMM fashion.

Simulation results and an application to the Fama-French 3-factor model show that the MDAPM can substantially reduce pricing errors compared to static Fama and MacBeth

(1973) regressions and the DAPM of Adrian et al. (2015). Filtered risk premia show a countercyclical pattern, with an initial decline at the beginning of crisis periods, and appear to differ crucially across factors.

Appendix

A Proofs

The following proofs of propositions use the following lemma. A proof of it can be found, for example, in Ronchetti and Trojani (2001) or Creal et al. (2024).

Lemma 1. *In case of the conditional moment condition* $\mathbb{E}_{t-1}\left[g_t(x_t;\vartheta_t,\tilde{\theta}_0)\right] = 0$*, the influence function in* (16) *is given by*

$$s_t = -\left(\bar{G}_{\vartheta,t}^\top \Omega_{t-1}\bar{G}_{\vartheta,t}\right)^{-1}\bar{G}_{\vartheta,t}^\top \Omega_{t-1} g_t(x_t;\vartheta_t,\tilde{\theta}_0) \tag{A.1}$$

with

$$\bar{G}_{\vartheta,t} = \mathbb{E}_{t-1}^{\epsilon}\left(\frac{\partial g_t(x_t;\vartheta_t,\tilde{\theta}_0)}{\partial \vartheta_t^\top}\right)\Bigg|_{\epsilon=0} \tag{A.2}$$

where $\mathbb{E}_{t-1}^{\epsilon}$ *is the conditional expectation based on the measure* F_x^{ϵ}*.*

A.1 Proof of Proposition 1

(a) For $\vartheta_t = \lambda_t$ we derive

$$\bar{G}_{\vartheta,t} = \mathbb{E}_{t-1}^{\epsilon}\left(\frac{\partial g_t(x_t;\vartheta_t,\theta_1)}{\partial \vartheta_t^\top}\right)\Bigg|_{\epsilon=0} = \begin{pmatrix} 0_{K\times K} & 0_{K\times NK_z} & 0_{K\times NK} & -\beta^\top \end{pmatrix}^\top \tag{A.3}$$

Using Lemma 1 and (A.3), we can derive the required influence function:

$$\begin{aligned} s_t = \frac{d\vartheta_t(F^{\epsilon})_x}{d\epsilon}\bigg|_{\epsilon=0} &= -\left(\bar{G}_{\vartheta,t}^\top \Omega_{t-1}\bar{G}_{\vartheta,t}\right)^{-1}\bar{G}_{\vartheta,t}^\top \Omega_{t-1} g_t(x_t;\vartheta_t,\theta_1) && \text{(A.4)}\\ &= \left(\beta^\top\beta\beta^\top\beta\right)^{-1}\beta^\top\beta\beta^\top e_t && \text{(A.5)}\\ &= \left(\beta^\top\beta\right)^{-1}\beta^\top e_t && \text{(A.6)}\\ &= \left(\beta^\top\beta\right)^{-1}\beta^\top r_t - \lambda_{t-1} - u_t. && \text{(A.7)} \end{aligned}$$

(b) Given $\vartheta_t = \left(\lambda^\top, vec(\tilde{\beta}_t)^\top\right)^\top$ we derive

$$\bar{G}_{\vartheta,t} = \begin{pmatrix} 0_{(NK_z+K)\times K} & 0_{(NK_z+K)\times NK} \\ 0_{NK\times K} & -\Sigma_u \otimes I_N \\ -\beta_t & -\lambda_t^\top \otimes I_N \end{pmatrix}. \tag{A.8}$$

We further find

$$\bar{G}_{\vartheta,t}^\top \Omega_{t-1} \bar{G}_{\vartheta,t} = \begin{pmatrix} 0_{K\times(NK_z+K)} & 0_{K\times NK} & -\beta_{t-1}^\top \\ 0_{NK\times(NK_z+K)} & -\Sigma_u \otimes I_N & -\lambda_t \otimes I_N \end{pmatrix} \begin{pmatrix} I_{M-N} & 0_{(M-N)\times N} \\ 0_{N\times(M-N)} & \beta_t\beta_t^\top \end{pmatrix}$$
$$\times \begin{pmatrix} 0_{(NK_z+K)\times K} & 0_{(NK_z+K)\times NK} \\ 0_{NK\times K} & -\Sigma_u \otimes I_N \\ -\beta_t & -\lambda_t^\top \otimes I_N \end{pmatrix} \tag{A.9}$$
$$= \begin{pmatrix} 0_{K\times(NK_z+K)} & 0_{K\times N} & -\beta_t^\top \beta_t \beta_t^\top \\ 0_{NK\times(NK_z+K)} & -\Sigma_u \otimes I_N & -\lambda_t \otimes \beta_t\beta_t^\top \end{pmatrix}$$
$$\times \begin{pmatrix} 0_{(NK_z+K)\times K} & 0_{(NK_z+K)\times NK} \\ 0_{NK\times K} & -\Sigma_u \otimes I_N \\ -\beta_t & -\lambda_t^\top \otimes I_N \end{pmatrix} \tag{A.10}$$
$$= \begin{pmatrix} \beta_t^\top \beta_t \beta_t^\top \beta_t & \lambda_t^\top \otimes \beta_t^\top \beta_t \beta_t^\top \\ \lambda_t \otimes \beta_t \beta_t^\top \beta_t & \Sigma_u^2 \otimes I_N + \lambda_t \lambda_t^\top \otimes \beta_t \beta_t^\top \end{pmatrix}. \tag{A.11}$$

Applying the block matrix Schur complement allows to invert matrix (A.11):

$$\left(\bar{G}_{\vartheta,t}^\top \Omega_{t-1} \bar{G}_{\vartheta,t}\right)^{-1} = \begin{pmatrix} (\beta_t^\top \beta_t)^{-2} + \lambda_t^\top \Sigma_u^2 \lambda_t (\beta_t^\top \beta_t)^{-1} & -\lambda_t^\top \Sigma_u^{-2} \otimes \beta_t (\beta_t^\top \beta_t)^{-1} \\ -\Sigma_u^{-2} \lambda_t \otimes (\beta_t^\top \beta_t)^{-1} \beta_t^\top & \Sigma_u^{-2} \otimes I_N \end{pmatrix}. \tag{A.12}$$

Moreover,

$$\bar{G}_{\vartheta,t}^{\top}\Omega_{t-1}g_t(x_t;\vartheta_t) = \begin{pmatrix} 0_{K\times(NK_z+K)} & 0_{K\times N} & -\beta_t^{\top}\beta_t\beta_t^{\top} \\ 0_{NK\times(NK_z+K)} & -\Sigma_u\otimes I_N & -\lambda_t\otimes\beta_t\beta_t^{\top} \end{pmatrix} \begin{pmatrix} u_t \\ \operatorname{vec}\left(u_t z_t^{\top}\right) \\ \operatorname{vec}\left(e_t u_t^{\top}\right) \\ e_t \end{pmatrix} \tag{A.13}$$

$$= \begin{pmatrix} -\beta_t^{\top}\beta_t\beta_t^{\top}e_t \\ -\operatorname{vec}\left(e_t u_t^{\top}\Sigma_u\right) - \lambda_t\otimes\beta_t\beta_t^{\top}e_t \end{pmatrix}. \tag{A.14}$$

Equations (A.11) and (A.14) together with Lemma 1 can be used to finally derive the influence function:

$$s_t = \begin{pmatrix} (\beta_t^{\top}\beta_t)^{-2} + \lambda_t^{\top}\Sigma_u^2\lambda_t(\beta_t^{\top}\beta_t)^{-1} & -\lambda_t^{\top}\Sigma_u^{-2}\otimes\beta_t(\beta_t^{\top}\beta_t)^{-1} \\ -\Sigma_u^{-2}\lambda_t\otimes(\beta_t^{\top}\beta_t)^{-1}\beta_t^{\top} & \Sigma_u^{-2}\otimes I_N \end{pmatrix}$$

$$\times \begin{pmatrix} -\beta_t^{\top}\beta_t\beta_t^{\top}e_t \\ -\operatorname{vec}\left(e_t u_t^{\top}\Sigma_u\right) - \lambda_t\otimes\beta_t\beta_t^{\top}e_t \end{pmatrix} \tag{A.15}$$

$$= \begin{pmatrix} \left(\beta_t^{\top}\beta_t\right)^{-1}\beta_t^{\top}e_t\left(1 - u_t^{\top}\Sigma_u^{-1}\lambda\right) \\ \operatorname{vec}\left(e_t u_t^{\top}\Sigma_u^{-1}\right) \end{pmatrix}. \tag{A.16}$$

□

A.2 Proof of Proposition 2

For $\vartheta_t = \left(\lambda_t^\top, vec(\beta_t)^\top, vech(\Sigma_{u,t})^\top, vech(\Sigma_{e,t})^\top\right)^\top$ we can derive

$$\bar{G}_{\vartheta,t} = \mathbb{E}_{t-1}^{\epsilon}\left(\frac{\partial g_t(x_t;\vartheta_t,\theta_1)}{\partial \vartheta_t^\top}\right)\bigg|_{\epsilon=0} \tag{A.17}$$

$$= \begin{pmatrix} 0_{(NK_z+K)\times K} & 0_{(NK_z+K)\times NK} & 0_{(NK_z+K)\times K(K+1)/2} & 0_{(NK_z+K)\times N(N+1)/2} \\ 0_{NK\times K} & -\Sigma_u \otimes I_N & 0_{NK\times K(K+1)/2} & 0_{NK\times N(N+1)/2} \\ -\beta_t & -\lambda_t^\top \otimes I_N & 0_{N\times K(K+1)/2} & 0_{N\times N(N+1)/2} \\ 0_{K(K+1)/2\times K} & 0_{K(K+1)/2\times NK} & I_{K(K+1)/2} & 0_{K(K+1)/2\times N(N+1)/2} \\ 0_{N(N+1)/2\times K} & 0_{N(N+1)/2\times NK} & 0_{N(N+1)/2\times K(K+1)/2} & I_{N(N+1)/2} \end{pmatrix}. \tag{A.18}$$

Similar derivations as in the proof of Proposition 1 yields to

$$\left(\bar{G}_{\vartheta,t}^\top \Omega_{t-1} \bar{G}_{\vartheta,t}\right)^{-1} = \begin{pmatrix} (\beta_t^\top\beta_t)^{-2} + \lambda_t^\top \Sigma_u^2 \lambda_t (\beta_t^\top \beta_t)^{-1} & -\lambda_t^\top \Sigma_u^{-2} \otimes \beta_t (\beta_t^\top\beta_t)^{-1} & 0_{K\times L} \\ -\Sigma_u^{-2}\lambda_t \otimes (\beta_t^\top\beta_t)^{-1}\beta_t^\top & \Sigma_u^{-2} \otimes I_N & 0_{N^2\times L} \\ 0_{L\times K} & 0_{L\times N^2} & I_L \end{pmatrix} \tag{A.19}$$

with $L = N(N+1)/2 + K(K+1)/2$ and

$$\bar{G}_{\vartheta,t}^\top \Omega_{t-1} g_t(x_t;\vartheta_t) = \begin{pmatrix} -\beta_t^\top \beta_t \beta_t^\top e_t \\ -\mathrm{vec}\left(e_t u_t^\top \Sigma_u\right) - \lambda_t \otimes \beta_t \beta_t^\top e_t \\ \mathrm{vech}\left(u_t u_t^\top - \Sigma_u\right) \\ \mathrm{vech}\left(e_t e_t^\top - \Sigma_e\right) \end{pmatrix}. \tag{A.20}$$

Equations (A.19) and (A.20) together with Lemma 1 prove the proposition.

□

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